Sigma Functions over Affine Plane Curves and their Jacobi Inversion Formulae

Shigeki Matsutani

Sagamihara & Canon Inc.

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Yokohama, JAPAN

Joint work with

Emma Previato,
Chris Eilbeck, Victor Enolskii, Yoshihiro Ônishi,

(EEMOP, I.M.R.Notices 07; MP, Israel M.J. 08; EEMOP, Crelle 08; EOM, J.Tokyo.M 08; MP, JMSJ 08)

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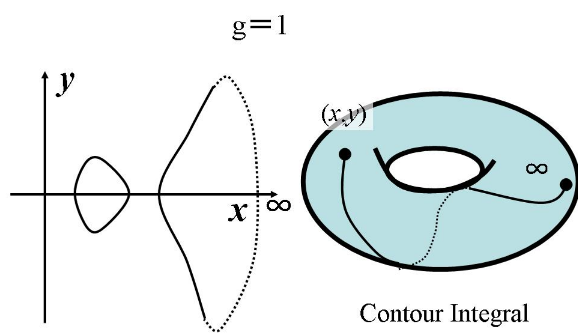
1. $g = 1$ curve: motivation

Let us consider an elliptic curve,

$$X_1 := \left\{ (x, y) \;\middle|\; \begin{aligned} y^2 &= x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ &= (x - e_1)(x - e_2)(x - e_3) \end{aligned} \right\} \cup \infty.$$



1. The Abelian integral $w : \text{Path}(X_1) \to \mathbb{C}$,
$$u = w(x, y) \equiv \int_{\infty}^{(x,y)} \nu^I, \quad \nu^I := \frac{dx}{2y},$$

2. The (half) periodicity (complete integral of the 1st kind):
$$\omega_i := w(e_i, 0) \equiv \int_{\infty}^{(e_i,0)} \nu^I, \quad \nu^I := \frac{dx}{2y},$$
where $(e_i, 0)$ $(i = 1, 2, 3)$ and ∞ are branch points.

3. The complete integral of the 2nd kind:
$$\eta_i := \int_{\infty}^{(e_i,0)} \nu^{II}, \quad \nu^{II} := \frac{x dx}{2y},$$

4. The lattice

$$\Lambda = 2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_3.$$

5. The Jacobi variety:

$$\mathcal{J} = \mathbb{C}/\Lambda,$$

6. Legendere relation

$$\omega_2\eta_1 - \omega_1\eta_2 = \frac{\pi}{2}.$$

7. The Weierstrass sigma function as an entire function over $\mathbb{C}$,

$$\sigma(u) = 2\omega_1 \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{\theta_1(\frac{u}{2\omega_1})}{\theta_1'^0}$$

(a) Translation formula for $\Omega_{m,n} := 2m\omega_1 + 2n\omega_3$:

$$\sigma(u+\Omega_{m,n}) = (-1)^{m+n+mn} \exp\left((m\eta_1 + n\eta_3)(2u + \Omega_{m,n})\right) \sigma(u).$$

(b) zeros of sigma:

$$\{\text{zeros} \quad \text{of} \quad \sigma\} \equiv 0 \quad \text{mod.} \quad \Lambda, \qquad \lim_{u\to 0} \frac{\sigma(u)}{u} = 1.$$

8. The Weierstrass $\wp$ function over $\mathcal{J} = \mathbb{C}/\Lambda$ is defined by

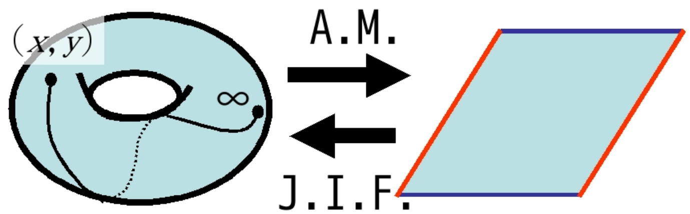
$$\zeta(u) = \frac{d}{du} \log \sigma(u). \qquad \wp(u) = -\frac{d^2}{du^2} \log \sigma(u).$$

9-a. Jacobi inversion formula

Theorem: (Jacobi inversion formula): For $u = w(x, y)$,

$$(x, y) = \left(\wp(u), \frac{1}{2}\wp_u(u)\right), \quad \wp_u(u) := \frac{d}{du}\wp(u).$$

$$X_1 \Leftrightarrow \mathcal{J}$$



$$(y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \;\Leftrightarrow\; \wp_u^2/4 = \wp^3 + \lambda_2 \wp^2 + \lambda_1 \wp + \lambda_0)$$

Remark The Jacobi inversion formula that $x = \wp(u)$, is non-trivial. By assuming it,

$$du = \frac{dx}{2y} \quad \text{implies that } \frac{d}{du} = 2y\frac{d}{dx}.$$

Hence $\frac{d}{du}x = 2y\frac{d}{dx}x = 2y$.

9-b. Due to the Jacobi inversion formula, we have a corollary

Corollary:

$$\text{“elliptic functions”} \equiv \begin{Bmatrix} \text{rational functions of x and y} \\ \text{whose arguments are u=w(x,y)} \end{Bmatrix}$$

$$x(u) - 1,\ y(u) - 2,\ x^2(u) + ax(u) + b,\ \frac{1}{1+x^2(u)},\ \frac{x}{y}(u),\ \frac{x^2(u)}{y(u)-a}\ \cdots.$$

$$\text{“trigonometric functions”} \equiv \begin{Bmatrix} \text{rational functions of x and y} \\ \text{whose arguments are } \theta \\ \text{of } x^2 + y^2 = 1 \end{Bmatrix}$$

10. Differential relations

1. $\wp_u^2 = 4(\wp^3 + \lambda_2\wp^2 + \lambda_1\wp + \lambda_0) \qquad (y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0)$.

2. $\wp_u\wp_{uu} = 6\wp^2\wp_u + 4\lambda_2\wp\wp_u + 2\lambda_1\wp_u$ or

$$\wp_{uu} - 6\wp^2 = 4\lambda_2\wp + 2\lambda_1.$$

3. $\wp_{uuu} = 12\wp\wp_u + 4\lambda_2\wp_u$ (KdV equation $v_t + 6vv_x + v_{xxx} = 0$).

   $u = \wp/2$.

11. Behaviour at ∞

$$y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \to y^2 \sim x^3 \text{ at } \infty.$$

Then the behaviour of x and y at ∞ is written by a local parameter t (6 is the least common multiple of 2 and 3).

$$x = \frac{1}{t^2}, \quad y = \frac{1}{t^3}(1 + o(t)), \quad \left(y^2 \sim x^3 \sim \frac{1}{t^6}\right)$$

On the other hand

$$du = \frac{dx}{2y} \sim \frac{-2/t^3 \cdot dt}{2/t^3(1 + o(t))} = (1 + o(t))dt$$

or

$$u = t(1 + o(t))$$

12. The commutative ring R, wt and r

Let $R = \mathbb{C}[x, y]/(y^2 - x^3 - \cdots)$ be identified with $\mathcal{O}_{X_1}(*\infty)$.

We define

1. the weight for an element ϕ of R:

   $$\mathrm{wt}(\phi) := \text{the order of singularity of } \phi \text{ at } \infty.$$

2. The order of the vanishing of u at the origin of $f \in \mathcal{O}(\mathbb{C})$

   $$\mathrm{r}(f) := \text{order of zero of } f \text{ at the origin.}$$

13. Weierstrass Gap Table

Table 1

wt/r	0	1	2	3	4	5	6	7	8	9	10
ϕ	1	-	x	y	x^2	xy	x^3	x^2y	x^4	x^3y	x^4y
$\langle u \rangle$	-	u	-	-	-	-	-	-	-	-	-

$\phi_0 = 1$, $\phi_1 = x$, $\phi_2 = y$, $\phi_3 = x^2$, $\phi_4 = xy$, $\phi_5 = x^3$, $\phi_6 = x^2y$, $\cdots$.

$$R = \oplus \mathbb{C}\phi_i \qquad \text{as a vector space}$$

"-" = gap for ϕ.

In the gap, there is a semigroup $H := \{2^a 3^b\}$ generated by $(2, 3)$.

$$H = \{0, 2, 3, 4, 5, 6, \cdots\}, \qquad L = \mathbb{Z} \setminus H = \{1\}.$$

14. Addition Formula: Frobenious-Stickelberger(1877)

For $u^{(i)} := w(x_i, y_i)$,

$$\frac{\sigma(u^{(0)} + u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)})}{\sigma^{n+1}(u^{(0)})\sigma^{n+1}(u^{(1)}) \cdots \sigma^{n+1}(u^{(n)})}$$

$$= C_n \begin{vmatrix} 1 & x_0 & y_0 & x_0 y_0 & \cdots & \phi_n(x_0, y_0) \\ 1 & x_1 & y_1 & x_1 y_1 & \cdots & \phi_n(x_1, y_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & y_n & x_n y_n & \cdots & \phi_n(x_n, y_n) \end{vmatrix}. \qquad (1)$$

1) σ function of addition of multivariables $u^{(i)} = w(x_i, y_i)$ is given by this determinant.

2) $\phi_j(x_i, y_i)$ comes from the gap-table.

15. Motivation

It should be recalled that

when we apply the elliptic functions theory to the other fields, e.g., number theory, physics, engineer, information theory and so on, their explicit formulas and relations play important roles.

The elliptic function theory provides the algebraic geometrical properties and the analytic properties.

In order to apply the theory of functions of curves with higher genera to the other fields in future, we would like to have the similar thoery and explicit relations and formulas of higher genus curves.

By considering the integrable nonlinear equations, KdV equation, the Neumann system, the sine-Gordon equations, Mumford constructed such theories in "Tata Lectures on Theta II" for every hyperelliptic curves based upon Jacobi's style, which is known as Mumford triplet U, V, W.

We would like to proceed the further study of Tata Theta II on evey affine plane curve based upon theories of Klein and Baker.

16. History of sigma functions

1. Abel, Jacobi discovered holomorphic one forms over a hyperelliptic curve.

2. Based upon the Jacobi sn, cn, dn functions, Weierstrass essentially discovered elliptic sigma functions 1841-2, which was called Al-function honor to Abel.

3. Jacobi discovered the Abel theorem among curve and Jacobian. Jacobi found Jacobi iversion formulae (which is the base of Mumford's Tata II).

4. 1856 Weierstrass discovered his theta function over every hyperellipci curve, Al function, which was, later sophiscated to Klein's σ function by Klein as a hyperelliptic version of the Weierstrass σ function.

5. 1857 Riemann discovered his Ablein function theory over every compact Riemann surface.

6. 1856-80 Weierstrass constructed his elliptic function theory, which was known via his lecture.

16-2. History of sigma functions

7. 1886 Klein defined the Kleinian σ function over every hyperelliptic surface.

8. 1896 Baker wrote a book on the Kleinian σ function over every hyperelliptic curve and other one.

9. 1903 Baker disovered the KdV hierarchy and KP equation over every hyperelliptic curve, which is known as the integrable nonlinear equations.

2. $g = 1$ Curve Revised: on sigma function)

1. $\wp(u) = -\dfrac{d^2}{du^2} \log \sigma$ means that we may find an entire function over $\mathbb{C}$,

$$\sigma(u) \sim \exp\left(\int^u \int^{u'} \wp(u'')du'du''\right).$$

We want to express the coordinate of the affine curve in terms of a generalized sigma function.

A meromorphic function h over $\mathcal{J}$ whose zeros are $(a_i)_{i=1,\cdots,N}$ and singularities are $(b_i)_{i=1,\cdots,N}$ is given by

$$h(u; a, b) = h_0 \mathrm{e}^{2(m\eta_1 + n\eta_2)u} \frac{\prod_{i=1}^N \sigma(u - a_i)}{\prod_{i=1}^N \sigma(u - b_i)}$$

where h_0 is a constant factor and $2m\omega_1 + 2n\omega_3 = \sum a_i - \sum b_i$.

2.

$$\wp(u-v)dudv \sim \frac{dudv}{(u-v)^2}, \quad \sigma(u-v) \sim (u-v), \quad \text{around} u \sim v$$

We may find the ratio of x's and y's to express $dudv/(u-v)^2$ and use the relation,

$$\int_{z_2}^{z_1}\int_{z_4}^{z_3} \frac{dw_1 dw_2}{2(w_1-w_2)^2} = \int_{z_2}^{z_1}\left(\frac{dw_1}{(w_1-z_3)} - \frac{dw_1}{(w_1-z_4)}\right)$$
$$= \log\left(\frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)}\right)$$

to characterize the sigma function.

3. For the curve $y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$, and $P_i = (x_1, y_1)$

$$u_i = \int_\infty^{(x_i, y_i)} \nu^I,$$

$$\begin{aligned}\Omega(P_1, P_2) &:= \wp(u_1 - u_2)\nu^I(P_1)\nu^I(P_2)\\ &= \frac{F(P_1, P_2) dx_1 \otimes dx_2}{(x_1 - x_2)^2 4 y_{P_1} y_{P_2}},\end{aligned}$$

where F is an element of $R \otimes R$,

$$\begin{aligned}F(P_1, P_2) &:= f(P_1, P_2) + y_1 y_2,\\ f(P_1, P_2) &:= \sum_{j=0}^{g} x_1^j x_2^j (\lambda_{2j+1}(x_1 + x_2) + 2\lambda_{2j})\\ &= x_1 x_2 (x_1 + x_2 + 2\lambda_2) + \lambda_1 (x_1 + x_2) + 2\lambda_0.\end{aligned}$$

4. **Fundamental differential of the 2nd kind:** We call the two-form $\Omega(P_1, P_2)$ on $X \times X$ **FDII**

   (a) Symmetric: $\Omega(P_1, P_2) = \Omega(P_2, P_1)$

   (b) only pole (of 2nd order) along the diagonal of $X \times X$, and in the vicinity of each point (P_1, P_2) is expanded in power series as

   $$\Omega(P_1, P_2) = \left(\frac{1}{(t_{P_1} - t'_{P_2})^2} + d_{\geq}(1)\right) dt_{P_1} \otimes dt_{P_2} \text{ (as } P_1 \to P_2),$$

   (c) $\Omega(P_1, P_2)$ is written only by the affine coordinaite.

5. **the Riemann fundamental relation:** Idea:

$$\int_{z_2}^{z_1}\int_{z_4}^{z_3}\frac{dw_1 dw_2}{2(w_1-w_2)^2}=\log\left(\frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)}\right)$$

Theorem (Riemann fundamental relation): For $(P, Q, P', Q') \in X^4$,

$$u := w(P), \quad u' := w(P)', \quad v := w(Q), \quad v' := w(Q'),$$

$$\exp\left(\int_Q^P\int_{Q'}^{P'}\Omega(P_1, P_2)\right)=\frac{\sigma(u-u')\sigma(v-v')}{\sigma(u-v')\sigma(v-u')}.$$

Applying $\partial/\partial u$ **and** $\partial/\partial u'$ **it, it recovers the relation** $\wp(u_1-u_2)$ **and Jacobi inversion formula.**

6. To determine the FDII, we find a way **(Klein, Bolza, and so on, Eilbeck-Enolskii-Leykin (EEL)**

   Let $\Sigma\Big(P,Q\Big)$ be the following form,

$$\Sigma\Big(P,Q\Big) := \frac{y_P + y_Q}{(x_P - x_Q)2y_P} dx_P. \tag{2}$$

7. Noting $\Sigma(P,Q)$ has the properties

   (a) $\Sigma(P,Q)$ as function of P has singular at Q, vanishes at $\iota_H(Q)$ and singular at ∞.

   (b) $\Sigma(P,Q)$ as function of Q has singular at P and at ∞.

7. **Lemma: ($\Omega(P_1, P_2)$ and $\Sigma(P_1, P_2)$)**

$$\Omega(P_1, P_2) = d_{P_2}\Sigma(P_1, P_2) + \nu^I(P_1) \otimes \nu^{II}(P_2)$$

where

$$d_Q\Sigma\Big(P, Q\Big) := dx_P \otimes dx_Q \frac{\partial}{\partial x_Q} \frac{y_P + y_Q}{(x_P - x_Q) 2y_P} dx_P. \qquad (3)$$

By the differentials of 2nd kind ν^{II} whose pole is at ∞, we remove the singularity of $d_Q\Sigma\Big(P, Q\Big)$ at ∞.

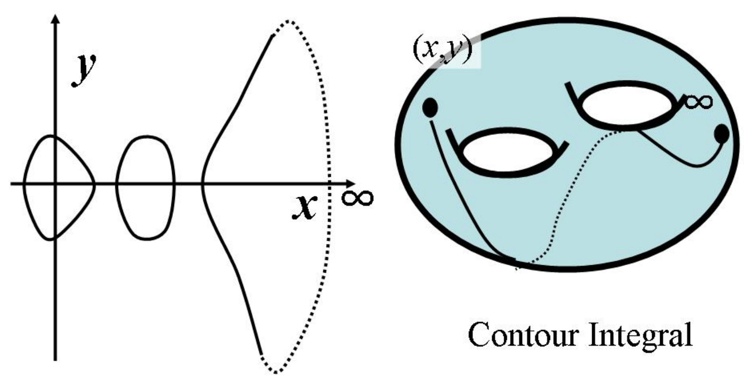
8. **Lemma: ($\Sigma(P_1, P_2)$ and ν^{II})**

$$\begin{aligned} d_Q\Sigma\Big(P, Q\Big) - d_P\Sigma\Big(Q, P\Big) &= \nu^I(Q) \otimes \nu^{II}(P) - \nu^I(P) \otimes \nu^{II}(Q) \\ &= \frac{dx_Q}{2y_Q} \otimes \frac{x_P dx_P}{2y_P} - \frac{dx_P}{2y_P} \otimes \frac{x_Q dx_Q}{2y_Q} \end{aligned} \qquad (4)$$

3. $g = 2$ Curve (of hyperelliptic curve)

$$X_2 : y^2 = x^5 + \lambda_4 x^4 + \cdots + \lambda_0$$

g=2



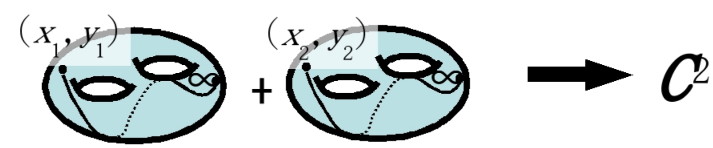
1. Abelian integral: $w : X_2 \to \mathbb{C}^2; \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = w(P)\right)$, $P = (x, y)$

$$w(P) := \left(\int_\infty^P \frac{dx}{2y}, \int_\infty^P \frac{xdx}{2y}\right)^t, \quad {\nu^I}_1 := \frac{dx}{2y}, \quad {\nu^I}_2 := \frac{xdx}{2y}.$$

${\nu^I}_a$ is the holormorphic one-form, differential of the 1st kind.

2. We have a correspondence

$$w : S^2 X_2 \to \mathbb{C}^2;$$
$$w(P_1, P_2) := w(P_1) + w(P_2).$$
$$\text{Path}(S^2(X_2)) \to \mathbb{C}^2 \text{ (surjection)}.$$



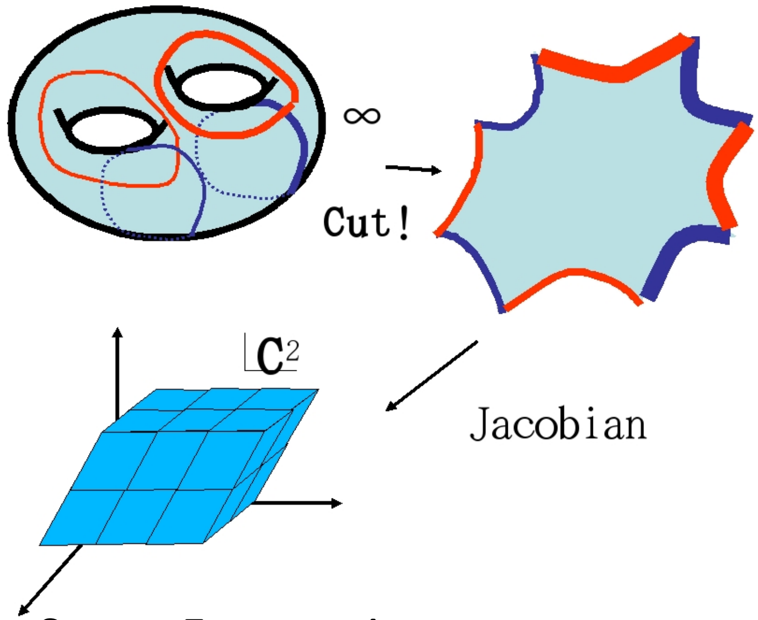
3. The w naturally defines the periodic (2×4) matrix ω and the lattice Λ:

$$\omega := \begin{pmatrix} \omega' & \omega'' \end{pmatrix},$$

$$\omega' := \frac{1}{2} \begin{pmatrix} \int_{\alpha_1} \nu^I{}_1 & \int_{\alpha_2} \nu^I{}_1 \\ \int_{\alpha_1} \nu^I{}_2 & \int_{\alpha_2} \nu^I{}_2 \end{pmatrix},$$

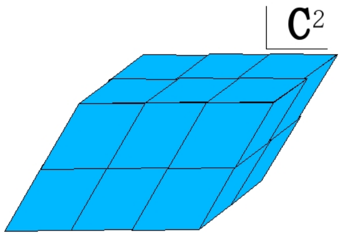
$$\omega'' := \frac{1}{2} \begin{pmatrix} \int_{\beta_1} \nu^I{}_1 & \int_{\beta_2} \nu^I{}_1 \\ \int_{\beta_1} \nu^I{}_2 & \int_{\beta_2} \nu^I{}_2 \end{pmatrix},$$

$$\Lambda := 2\omega \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \\ \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} \subset \mathbb{C}^2.$$

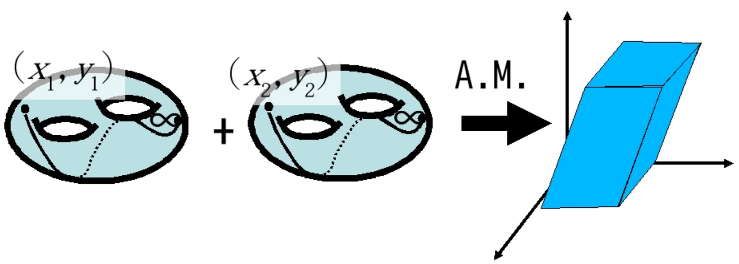


$$y^2 = x^5 + \lambda_4 x^4 + \cdots + \lambda_1 x + \lambda_0$$

4. $\kappa : \mathbb{C}^2 \to \mathcal{J} := \mathbb{C}^2/\Lambda$. $\mathcal{J}$ is the Jacobian.



5. The Abel theorem is that $\kappa \circ w : S^2 X_2 \to \mathcal{J}$ is birational.



6. **Realization of** $\Sigma(P_1, P_2)$ **:**

Let $\Sigma\Big(P, Q\Big)$ be the following form,

$$\Sigma\Big(P, Q\Big) := \frac{y_P + y_Q}{(x_P - x_Q)2y_P} dx_P. \tag{5}$$

Noting $\Sigma(P, Q)$ has the following properties

(a) $\Sigma(P, Q)$ as function of P has singular at $Q = (x_Q, y_Q)$, vanishes at $\iota_H(Q) = (x_Q, -y_Q)$ and singular at ∞.

(b) $\Sigma(P, Q)$ as function of Q has singular at P and at ∞.

7. **Realization of differentials of the 2nd kind ν^{II} :**

The differentials of 2nd kind ${\nu^{II}}_j$ $(j = 1, 2)$ whose pole is at ∞;

$$
\begin{aligned}
&d_Q\Sigma\Big(P,Q\Big) - d_P\Sigma\Big(Q,P\Big) \\
&\qquad = \sum_{i=1}^{2}\Big({\nu^{I}}_i(Q)\otimes{\nu^{II}}_i(P) - {\nu^{I}}_i(P)\otimes{\nu^{II}}_i(Q)\Big)
\end{aligned}
\tag{6}
$$

Then the differential of the 2nd kind is given by

$$
{\nu^{II}}_1 = \frac{3x^3 + 2\lambda_4 + 3\lambda_3}{2y}, \quad {\nu^{II}}_2 = \frac{x^2}{2y}.
$$

9. **The fundamental differential of the 2nd kind** $\Omega(P_1, P_2)$ in terms of the coordinate of hyperelliptic curve

$$\begin{aligned}\Omega(P_1,P_2) &= d_{P_2}\Sigma(P_1,P_2) + \sum_{i=1}^{g} \nu^{I}{}_i(P_1) \otimes \nu^{II}{}_i(P_2) \\ &= \frac{F(P_1,P_2)dx_1 \otimes dx_2}{(x_1-x_2)^2 4y_{P_1}y_{P_2}},\end{aligned} \tag{7}$$

where F is an element of $R \otimes R$,

$$F(P_1,P_2) = f(P_1,P_2) + y_1 y_2,$$

$$f(P_1,P_2) = \sum_{j=0}^{g} x_1^j x_2^j (\lambda_{2j+1}(x_1+x_2) + 2\lambda_{2j}).$$

10. **Proposition (Legendre relation) :** For the complete integral of the 2nd kind is given by

$$\eta' := \frac{1}{2}\begin{pmatrix} \int_{\alpha_1} \nu^{II}{}_1 & \int_{\alpha_2} \nu^{II}{}_1 \\ \int_{\alpha_1} \nu^{II}{}_2 & \int_{\alpha_2} \nu^{II}{}_2 \end{pmatrix}, \quad \eta'' := \frac{1}{2}\begin{pmatrix} \int_{\beta_1} \nu^{II}{}_1 & \int_{\beta_2} \nu^{II}{}_1 \\ \int_{\beta_1} \nu^{II}{}_2 & \int_{\beta_2} \nu^{II}{}_2 \end{pmatrix}.$$

and the Legendre relations is given by

$$\eta' \omega''^t - \eta'' \omega'^t = \frac{\pi}{2} I_{2\times 2}.$$

11. **Definition (Kleinain σ function) :** Klein defined Kleinian σ function as an extension of the Weierstrass σ function (1886,8), which is a nice θ function is defined by

$$\sigma(u) := \gamma_0 \mathrm{e}^{-u^t \omega'^{-1} {}^t\eta' u} \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (\omega'^{-1} u; \omega'^{-1} \omega''),$$

where γ_0 is constant, δ's are theta characteristics,

$$\delta' = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \quad \delta'' = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$

and **the Riemann theta function** is given by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z; \tau) = \sum_{n \in \mathbb{Z}^2} \exp\left(\pi\sqrt{-1}((n+a)^t \tau (n+a) - (n+a)^t (z+b))\right).$$

12. **Proposition: properties of σ:**

(a) **translational forumlae:** For u, $v \in \mathbb{C}^g$, and ℓ $(= 2\omega'\ell' + 2\omega''\ell'') \in \Lambda$, we define

$$L(u, v) := 2\ {}^t u(\eta' v' + \eta'' v''),$$

$$\chi(\ell) := \exp[\pi\sqrt{-1}\Big(2({}^t\ell'\delta'' - {}^t\ell''\delta') + {}^t\ell'\ell''\Big)]\ (\in \{1, -1\}).$$

The following holds

$$\sigma(u + \ell) = \sigma(u) \exp(L(u + \frac{1}{2}\ell, \ell))\chi(\ell). \tag{8}$$

(b) **zeros of σ:**

$$\{\text{zeros} \quad \text{of} \quad \sigma\} = \kappa^{-1}\Theta,$$

where the θ divisor Θ is given by $\kappa \circ w : X_2 \to \Theta \subset \mathcal{J}$.

13. **Theorem (The Riemann fundamental relation):** For $(P, Q, P_i, P_i') \in X^2 \times (S^2(X) \setminus S_1^2(X)) \times (S^2(X) \setminus S_1^2(X))$,

$$u := \sum_{i=1}^{2} w(P_i), \quad v := \sum_{i=1}^{2} w(P_i'),$$

$$\exp\left(\sum_{i,j=1}^{2} \Omega_{P_i, P_j'}^{P,Q}\right) = \frac{\sigma(w(P) - u)\sigma(w(Q) - v)}{\sigma(w(Q) - u)\sigma(w(P) - v)}$$

$$= \frac{\sigma(w(P) - w(P_1, \cdots, P_4))\sigma(w(Q) - w(P_1', \cdots, P_4'))}{\sigma((w(Q) - w(P_1, \cdots, P_4))\sigma(w(P) - w(P_1', \cdots, P_4'))}$$

14. **Definition: (Kleinian $\wp$ functions and ζ functions)**

(a) The Kleinian $\wp$ functions

$$\wp_{ij} := -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma \quad \in \Gamma(\mathcal{J}, \mathcal{O}(2\Theta)).$$

(b) The Kleinian ζ functions

$$\zeta_i := \frac{\partial}{\partial u_i} \log \sigma$$

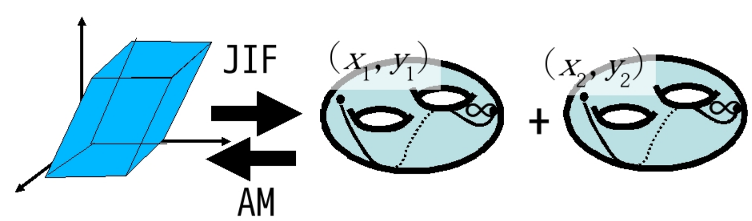
15. **Jacobi inversion formula** (Jacobi 1846)

Theorem: (Jacobi 1846) For $((x_1, y_1), (x_2, y_2)) \in S^2X_2$ and $u := w((x_1, y_1), (x_2, y_2)) \in \mathbb{C}^2$,

$$x^2 + \wp_{22}(u)x + \wp_{21}(u) = x^2 - (x_1 + x_2)x + x_1x_2$$
$$= \frac{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x & x^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}}.$$

This implies

$$\mathcal{J} \equiv \kappa w(S^2X_2) \to S^2X_2,$$



16. **Corollary:**

$$\text{“hyperelliptic functions”} \equiv \left\{ \begin{array}{c} \text{rational functions of} \\ \text{symmetric functions} \\ \text{of } x_1, y_1, x_2, y_2 \text{ whose arguments} \\ \text{are } u = w((x_1, y_1), (x_2, y_2)) \end{array} \right\}$$

$$(x_1 x_2)(u),\ (x_1 + x_2)(u),\ (y_1 + y_2)(u),\ \frac{x_1 x_2}{y_1 + y_2}(u),\ \cdots$$

17. **Jacobi inversion formula over a strata**

Theorem: (Grant 1988) The stratification of Jacobian:

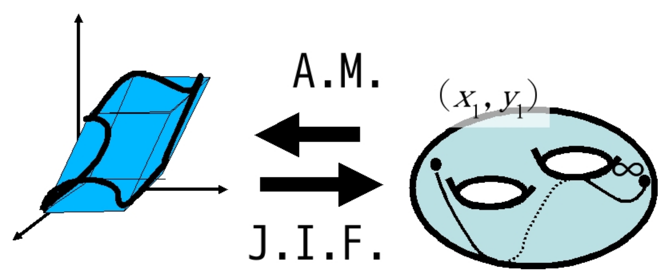
$$\{0\} \subset W^1 \subset \mathcal{J}, \quad W^1 := \kappa w(X_2), \quad \mathcal{J} \equiv \kappa w(S^2 X_2).$$

For $u = w(x_1, y_1) \in \mathbb{C}^2$,

$$x + \frac{\sigma_1}{\sigma_2}(u) = \frac{\begin{vmatrix} 1 & x_1 \\ 1 & x \end{vmatrix}}{\begin{vmatrix} 1 \end{vmatrix}} = x - x_1, \quad i.e., \quad x_1 = \frac{\sigma_1}{\sigma_2}(u),$$

where $\sigma_i = \partial\sigma/\partial u_i$.

$$W^1 \equiv \kappa w(X_2) \rightleftharpoons X_2,$$



18. **Theorem: Differential identical relations:**

KdV hierarchy (Baker 1903)

$$\wp_{2222} - 6\wp_{22}^2 = \frac{1}{8}\lambda_3 + \lambda_4\wp_{22} + \wp_{21}$$

$$\wp_{2221} - 6\wp_{22}\wp_{21} = \lambda_4\wp_{21} - \frac{1}{2}\wp_{11}$$

$$\wp_{2211} - 4\wp_{21}^2 - 2\wp_{22}\wp_{11} = \frac{1}{2}\lambda_3\wp_{21}$$

$$\wp_{2111} - 6\wp_{22}\wp_{12} = -\frac{1}{4}\lambda_0 - \frac{1}{2}\lambda_1\wp_{22} + \lambda_2\wp_{21}$$

$$\wp_{1111} - 6\wp_{11} = -\frac{1}{4}\lambda_0\lambda_4 + \frac{1}{8}\lambda_1\lambda_3 - 3\lambda_0\wp_{22} + \lambda_1\wp_{21} + \lambda_2\wp_{11}$$

(KdV equation: $v_t + 6vv_x + v_{xxx} = 0, \quad t = v_1 - \lambda_4 v_2, x = v_2, v = \wp_{22}/2$

19. **Differential identical relation II: dispersionless KdV equation (MP09)**

Theorem: (MP09) For $u = w(x,y) \in \mathbb{C}^2$,

$$\frac{\partial}{\partial u_1} x = x \frac{\partial}{\partial u_2} x.$$

proof: due to the relation $2y\frac{d}{dx}x = x\frac{2y}{x}\frac{d}{dx}x$. ($du_1 = \frac{dx}{2y}$, $du_2 = \frac{xdx}{2y}$.)

KdV equation: $v_t + 6vv_x + v_{xxx} = 0$.

dKdV equation: $v_t + 6vv_x = 0$.

20. **Behaviour at ∞**

$$y^2 = x^5 + \lambda_4 x^4 + \cdots + \lambda_1 x + \lambda_0 \to y^2 \sim x^5 \text{ at } (\infty, \infty) \equiv \infty.$$

Then the behaviour of x and y at ∞ is written by a local parameter t (10 is the least common multiple of 2 and 5).

$$x = \frac{1}{t^2}, \quad y = \frac{1}{t^5}(1 + o(t)), \quad \left(y^2 \sim x^5 \sim \frac{1}{t^{10}}\right)$$

On the other hand $du_1 = \frac{dx}{2y}$ and $du_2 = \frac{xdx}{2y}$,

$$du_1 \sim \frac{-2/t^3 \cdot dt}{2/t^5(1 + o(t))} = t^2(1+o(t))dt, \quad du_2 \sim \frac{-2/t^3 \cdot 1/t^2 \cdot dt}{2/t^5(1 + o(t))} = (1+o(t))dt$$

or

$$u_1 = t^3(1 + o(t)), \quad u_2 = t(1 + o(t))$$

21. **Weierstrass Gap Table**

The distribution of wt and r of $y^2 = x^5 + \cdots$ are given by the Table 2:

Table 2

wt/r	0	1	2	3	4	5	6	7	8	9	10
ϕ	1	-	x	-	x^2	y	x^3	xy	x^4	x^2y	x^5
$\langle u_1, u_2 \rangle$	-	u_2	-	u_1	-	-	-	-	-	-	-
	3	2	1	0							

$\phi_0 = 1$, $\phi_1 = x$, $\phi_2 = x^2$, $\phi_3 = y$, $\phi_4 = x^2$, $\phi_5 = xy$, $\phi_6 = x^3$, $\cdots$.

$$R = \mathbb{C}[x, y]/(y^2 - x^5 - \cdots) = \oplus \mathbb{C}\phi_i \qquad \text{as a vector sp.}$$

"-" $=$ gap for ϕ.

In the gap, there is a semigroup $H := \{2^a 5^b\}$ generated by $(2, 5)$.

$$H = \{0, 2, 4, 5, 6, \cdots\}, \qquad L = \mathbb{Z} \setminus H = \{1, 3\}.$$

22. **Addition Formula: Frobenious-Stickelberger Relation**

Therom (Ônishi 1997) For $u^{(i)} := w(x_i, y_i)$,

$$\frac{\sigma(u^{(0)} + u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)})}{\sigma^{n+1}(u^{(0)})\sigma^{n+1}(u^{(1)}) \cdots \sigma^{n+1}(u^{(n)})}$$

$$= c_n \begin{vmatrix} 1 & x_0 & x_0^2 & y_0 & \cdots & \phi_n(x_0, y_0) \\ 1 & x_1 & x_1^2 & y_1 & \cdots & \phi_n(x_1, y_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & y_n & \cdots & \phi_n(x_n, y_n) \end{vmatrix}. \tag{9}$$

1) σ function of addition of multivariables $u^{(i)} = w(x_i, y_i)$ is given by this determinant.

2) $\phi_j(x_i, y_i)$ comes from the gap-table.

23. **Summary:** $g = 2$ **hyperelliptic case**

(a) The formulas are simple, explicit and concrete like ones of elliptic functions.

(b) Even though σ function is a transcendent with respect to u, its behaviours are algebraic, e.g., it generates meromorphic functions of coordinates of the plane curves.

24. **Fact:: These relations were generalised to every hyper-elliptic curve.**

(Klein 1886,8, Bolza 1895, Baker 1903, Buchsterber-Enolskii-Leykin, 1997, Ônishi (Iwate) 2005, Enolski-Gibbons 2002, Previato-M 2008)

Hyperelliptic curve is given by

$$y^2 = x^{2g+1} + \lambda_{2g} x^{2g} + \cdots + \lambda_1 x + \lambda_0.$$

General genus g hyperelliptic curve:

24-a Theorem: Jacobi inversion formula (Klein 1886,8, Baker 1897)

$$\wp_{ij} = \frac{\partial^2}{\partial u_i \partial u_j} \log \sigma.$$

$$\mathcal{J} = \kappa w(S^g X_g) \text{ case: } u = w((x_i, y_i)_{i=1,\cdots,g}).$$

$$x^g + \sum_{i=0}^{g-1} \wp_{gg-i}(u) x^i = \frac{\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^g \\ 1 & x_2 & x_2^2 & \cdots & x_2^g \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_g & x_g^2 & \cdots & x_g^g \\ 1 & x & x^2 & \cdots & x^g \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{g-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{g-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_g & x_g^2 & \cdots & x_g^{g-1} \end{vmatrix}}$$

$\wp_{ij}$ is elementary symmetric function of $x_1, \cdots, x_g$.

General genus g hyperelliptic curve:

24-b Theorem: Jacobi inversion formulae

(Enolskii 2002(unpublished), MP 2008)

The stratification of Jacobian:

$$\{0\} \subset W^1 \subset W^2 \subset \cdots \subset \mathcal{J} \equiv W^g, \quad W^k := \kappa w(S^k X_2).$$

$W^k := \kappa w(S^k X_g)$ case ($k < g$) For $u = w((x_i, y_i)_{i=1,\cdots,k})$

$$x^k + \sum_{i=0}^{k-1} \frac{\sigma_{k-i+1}}{\sigma_{k+1}}(u) x^i = \frac{\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^k \\ 1 & x & x^2 & \cdots & x^k \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} \end{vmatrix}}, \quad , \frac{\sigma_1}{\sigma_2}(u) = x_1.$$

where $\sigma_i = \partial \sigma / \partial u_i$.

24-c Thoerem: Addition Formula (EEMOP 07)

For $u := \sum^m w(x_i, y_i)$, and $v := \sum^m w(x'_i, y'_i)$ $(m, n \geq g)$,

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)} = \delta(g,m,n)$$
$$\times \frac{\prod_{i=0}^{1} \Delta_{m+n}((x_1,y_1),\cdots,(x_m,y_m),(x'_1,(-1)^i y'_1),\cdots,(x'_n,(-1)^i y'_n))}{(\Delta_m((x_1,y_1)\cdots,(x_m,y_m))\Delta_n((x'_1,y'_1)\cdots,(x'_n,y'_n)))^2}$$
$$\times \prod_{i=1}^{m}\prod_{j=1}^{n} \frac{1}{\Delta_2((x_i,y_i),(x'_j,y'_j))}$$

where $\delta(g,m,n) = (-1)^{gn+\frac{1}{2}n(n-1)+mn}$.

$$\Delta_n((x_1,y_1),\cdots,(x_n,y_n))$$
$$= \begin{vmatrix} 1 & \phi_1(x_1,y_1) & \phi_2(x_1,y_1) & \cdots & \phi_{n-2}(x_1,y_1) & \phi_{n-1}(x_1,y_1) \\ 1 & \phi_1(x_2,y_2) & \phi_2(x_2,y_2) & \cdots & \phi_{n-2}(x_2,y_2) & \phi_{n-1}(x_2,y_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_1(x_{n-1},y_{n-1}) & \phi_2(x_{n-1},y_{n-1}) & \cdots & \phi_{n-2}(x_{n-1},y_{n-1}) & \phi_{n-1}(x_{n-1},y_{n-1}) \\ 1 & \phi_1(x_n,y_n) & \phi_2(x_n,y_n) & \cdots & \phi_{n-2}(x_n,y_n) & \phi_{n-1}(x_n,y_n) \end{vmatrix}$$

24-d Summary : every hyperelliptic curve

The following properties preserves even for every hyperelliptic curve.

1. The formulas are simple, explicit and concrete like ones of elliptic functions.

2. Even though σ function is a transcendent with respect to u, its behaviours are algebraic, e.g., it generates meromorphic functions of coordinates of the plane curves.

3. $g = 3$ Curve (of non-hyperelliptic curve)

Purpose II

We want to deal with algebraic functions over every non-hyperelliptic plane curve like ones over hyperelliptic curves.

We will consider the genus $g = 3$ curve as the first step.

Non-hyperelliptic $g = 3$ case

$$X_3 : y^3 = x^4 + \lambda_3 x^3 + \cdots + \lambda_0 \text{ and } g = 3,$$

1. Abelian integral: $w : \text{Path}(X_3) \to \mathbb{C}^3$, $w(P) := \int_\infty^P \begin{pmatrix} {\nu^I}_1 \\ {\nu^I}_2 \\ {\nu^I}_2 \end{pmatrix}$;

$$ {\nu^I}_1 := \frac{dx}{3y^2}, \quad {\nu^I}_2 := \frac{xdx}{3y^2}, \quad {\nu^I}_3 := \frac{dx}{3y}.$$

2. Extended to its domain to $S^a(X_3)$ $(a = 1, 2, 3)$ by

$$w : \text{Path}(S^a X_3) \to \mathbb{C}^3; \quad w(P_1, \cdots, P_a) = \sum_{i=1}^{n} w(P_i).$$

3. The w naturally define the periodic matrix and the lattice $\Lambda \in \mathbb{C}^3$.

$$\kappa \mathbb{C}^3 \to \mathcal{J} := \mathbb{C}^3/\Lambda$$

4. The Abel theorem is that $\kappa \circ w : S^3 X_3 \to \mathcal{J}$ is birational.

5. Buchsterber-Enolskii-Leykin defined Kleinian σ function over X_3 as an extension of the Weierstrass σ function (BEL 1999, 2000, EEL2000).

6. The Kleinian $\wp$ functions

$$\wp_{ij} := \frac{\partial}{\partial u_i \partial u_j} \log \sigma.$$

7. Eilbeck-Enolskii-Leykin (EEL) constuction:

$$\Sigma\Big(P,Q\Big) := \frac{y_P^2 + y_P y_Q + y_Q^2}{(x_P - x_Q)3y_P^2} dx_P. \tag{10}$$

Noting $\Sigma(P,Q)$ has the properties:

(a) $\Sigma(P,Q)$ as function of P has singular at Q, vanishes at $\iota_H(Q)$ and singular at ∞.

(b) $\Sigma(P,Q)$ as function of Q has singular at P and at ∞.

8. The differentials of the 2nd kind:

$$\nu^{II}{}_1 = \frac{-(5x^2 - 3\lambda_3 x + \lambda_6) y dx}{3y^2}, \quad \nu^{II}{}_2 = \frac{-2xydx}{3y^2}, \quad \nu^{II}{}_3 = \frac{-x^2 dx}{3y^2}.$$

9. Weierstrass Gap Table

$$X_3 : y^3 = x^4 + \lambda_3 x^3 + \cdots + \lambda_0$$

Table 3

wt/r	0	1	2	3	4	5	6	7	8	9	10
ϕ	1	-	-	x	y	-	x^2	xy	y^2	x^3	x^2y
$\langle u \rangle$	-	u_3	u_2	-	-	u_1	-	-	-	-	-
	5	4	3	2	1	0					

$$\phi_0 = 1,\ \phi_1 = x,\ \phi_2 = y,\ \phi_3 = x^2,\ \phi_4 = xy,\ \cdots$$

In the gap, there is a semigroup $H := \{3^a 4^b\}$ generated by $(3, 4)$.

$$H = \{0, 3, 4, 6, \cdots\}, \quad L = \mathbb{Z} \setminus H = \{1, 2, 5\}.$$

10. Jacobi inversion formula

(Enolskii Eilbeck Leykin 2000, Buchsterber Enolskii Leykin 1999)

Thereom: For $u = w((x_1, y_1), (x_2, y_2), (x_3, y_3))$,

$$x^2 + \wp_{33}(u)y + \wp_{32}(u)x + \wp_{31}(u) = \frac{\begin{vmatrix} 1 & x_1 & y_1 & x_1^2 \\ 1 & x_2 & y_2 & x_2^2 \\ 1 & x_3 & y_3 & x_3^2 \\ 1 & x & y_1 & x^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}$$

$$\mathcal{J} \rightleftharpoons S^3 X_3$$

11. Jacobi inversion formulae (Previato-M 2008)

Thereom: Let us consider the stratification of Jacobian:

$$\{0\} \subset W^1 \subset W^2 \subset \mathcal{J} \equiv W^3, \quad W^k := \kappa w(S^k X_3).$$

For $u = w((x_1, y_1), (x_2, y_2))$,

$$y + \frac{\sigma_2}{\sigma_3}(u)x + \frac{\sigma_1}{\sigma_3}(u) = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix},$$

where $\sigma_i = \partial\sigma/\partial u_i$.

For $u = w((x_1, y_1))$,

$$x + \frac{\sigma_1}{\sigma_2}(u) = \begin{vmatrix} 1 & x_1 \\ 1 & x \end{vmatrix}.$$

12. Differential relations

(Eilbeck, Enolskii, Leykin 2000), (EEMOP2007)

$$\wp_{3333} = 6\wp_{33}^2 + \wp_{22}$$
$$\wp_{2333} = 6\wp_{32}\wp_{33} + 3\lambda_3\wp_{33}$$
$$\wp_{2233} = 6\wp_{32}^2 + 2\wp_{33}\wp_{22} + 2\lambda_2 + 4\wp_{13}$$
$$\wp_{2223} = 6\wp_{22}\wp_{23} + 3\lambda_3\wp_{22}$$
$$\wp_{2222} = 6\wp_{22}^2 + 24\wp_{13}\wp_{33} - 3\lambda_3^2\wp_{33} + 12\lambda_2\wp_{33}$$

. . . .

. . . .

. . . .

13. Differential relations II

(dKdV equation:) For $u \in w(X_2) \subset \mathbb{C}^2$,

$$\frac{\partial}{\partial u_1} x = x \frac{\partial}{\partial u_3} x.$$

it is proved by the fact $2y \frac{d}{dx} x = x \frac{2y}{x} \frac{d}{dx} x$.

4. $y^r = f(x)$ Curve (C_{ab} curve)

Purpose III:

We want to deal with algebraic functions over every non-hyperelliptic plane curve $g > 3$ like ones over hyperelliptic curves.

1. General cyclic plane curve $X^g : y^r = x^s + \cdots + \lambda_0$:

Let us consider a more general plane curve of cyclic (r, s) type

$$X^g := \{(x, y) \mid y^r = x^s + \cdots + \lambda_0\} \cup \infty.$$

where r and s are positive integers of $r < s$ and $(r, s) = 1$.

2. Commutative Ring:

Let $R := \mathbb{C}[x, y]/(y^r - x^s - \cdots - \lambda_0)$ and ϕ_i be a monomial in R such that $wt(\phi_{i-1}) < wt(\phi_i)$, $\phi_0 = 1$ and

$$R = \oplus_{i=0}^{\infty} \mathbb{C}\phi_i.$$

3. Jacobi inversion formulae (Previato-M 2008)

$\mathcal{J} = \kappa w(S^g X_g)$ case: $u = w((x_1, y_1), \cdots, (x_g, y_g)) \in \mathbb{C}^g$

$$\phi_g(P) + \sum_{i=0}^{g-1} \wp_{gg-i}(u)\phi_i(P) = \frac{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_g(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_g(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_g) & \phi_2(P_g) & \cdots & \phi_g(P_g) \\ 1 & \phi_1(P) & \phi_2(P^2) & \cdots & \phi_g(P) \end{vmatrix}}{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_{g-1}(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_{g-1}(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_g) & \phi_2(P_g) & \cdots & \phi_{g-1}(P_g) \end{vmatrix}}$$

General plane curve $X^g : y^r = x^s + \cdots + \lambda_0$:

4. Jacobi inversion formulae (Previato-M 2008)

The stratification of Jacobian:

$$\{0\} \subset W^1 \subset W^2 \subset \cdots \subset \mathcal{J} \equiv W^g, \quad W^k := \kappa w(S^k X_2).$$

$W^k := \kappa w(S^k X_g)$ case $(i < g)$

$$\phi_k(P) + \sum_{i=0}^{k-1} \frac{\sigma_{k-i+1}}{\sigma_{k+1}} x^i \phi_i(P) = \frac{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_k(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_k(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_g) & \phi_2(P_g) & \cdots & \phi_k(P_g) \\ 1 & \phi_1(P) & \phi_2(P^2) & \cdots & \phi_k(P) \end{vmatrix}}{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_{k-1}(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_{k-1}(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_g) & \phi_2(P_g) & \cdots & \phi_{k-1}(P_g) \end{vmatrix}}$$

5. Summary : $C_{r,s}$ curve

The following properties preserves even for every curve of $y^r = x^s + \lambda_{s-1}x^{s-1} + \cdots$.

1. The formulas are simple, explicit and concrete like ones of elliptic functions.

2. Even though σ function is a transcendent with respect to u, its behaviours are algebraic, e.g., it generates meromorphic functions of coordinates of the plane curves.

5a. Projects and Problems

1. Revise the Riemann-Kempf singularity theorem in terms of this theoretical framework. (Almost done!) (M-Previato 2009)

2. Define the sigma function over a space curve, extend the Jacobi iversion formula (Almost done!) (Komeda-M-Previato 2009), and investigate their properties.

3. As the sigma function is modular invariant, use its properties and consider the moduli space (partially started by Buchstaber and Leykin 2003)!

5b. Projects and Problems

4. As we have the Frobenious-Stickelberger for the cyclic curves, extend it to more general curve with Galois action.

5. As the structure of the addition formula of hyperelliptic curve is revealed well, write it down for general curves.

6. Extend every relation in the elliptic curve to one for more general curve.

7. Apply them to physical and number theoretic problems.

Thanks!