

A Short Note on Sigma Functions over Affine Plane Curves and their Jacobi Inversion Formulae

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Abstract

I give an overview of the recent progress on the problem written in the title as joint work with Emma Previato, Chris Eilbeck, Victor Enolskii, Yoshihiro Ônishi, [EEMOP07, EEMOP08, EP08, EP09a, EP09b, EMO]

1 $g = 1$ curve: motivation

Let us consider an elliptic curve,

$$X_1 := \left\{ (x, y) \mid \begin{array}{l} y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ = (x - e_1)(x - e_2)(x - e_3) \end{array} \right\} \cup \infty.$$

1. The Abelian integral $w : \text{Path}(X_1) \rightarrow \mathbb{C}$,

$$u = w(x, y) \equiv \int_{\infty}^{(x,y)} \nu^I, \quad \nu^I := \frac{dx}{2y},$$

2. The (half) periodicity (complete integral of the 1st kind):

$$\omega_i := w(e_i, 0) \equiv \int_{\infty}^{(e_i,0)} \nu^I, \quad \nu^I := \frac{dx}{2y},$$

where $(e_i, 0)$ ($i = 1, 2, 3$) and ∞ are branch points.

3. The complete integral of the 2nd kind:

$$\eta_i := \int_{\infty}^{(e_i,0)} \nu^{II}, \quad \nu^{II} := \frac{x dx}{2y},$$

4. The lattice

$$\Lambda = 2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_3.$$

5. The Jacobi variety:

$$\mathcal{J} := \mathbb{C}/\Lambda.$$

6. Legendre relation

$$\omega_2 \eta_1 - \omega_1 \eta_2 = \frac{\pi}{2}.$$

7. The Weierstrass sigma function as an entire function over \mathbb{C} ,

$$\sigma(u) = 2\omega_1 \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{\theta_1\left(\frac{u}{2\omega_1}\right)}{\theta_1^0}$$

- (a) translation formula for $\Omega_{m,n} := 2m\omega_1 + 2n\omega_3$:

$$\sigma(u + \Omega_{m,n}) = (-1)^{m+n+mn} e^{((m\eta_1 + n\eta_3)(2u + \Omega_{m,n}))} \sigma(u).$$

- (b) zeros of sigma:

$$\{\text{zeros of } \sigma\} \equiv 0 \pmod{\Lambda}, \quad \lim_{u \rightarrow 0} \frac{\sigma(u)}{u} = 1.$$

8. The Weierstrass \wp function over $\mathcal{J} = \mathbb{C}/\Lambda$ is defined by

$$\zeta(u) = \frac{d}{du} \log \sigma(u), \quad \wp(u) = -\frac{d^2}{du^2} \log \sigma(u).$$

9. Jacobi inversion formula: **Theorem: (Jacobi inversion formula):** For $u = w(x, y)$,

$$(x, y) = \left(\wp(u), \frac{1}{2} \wp_u(u) \right), \quad \wp_u(u) := \frac{d}{du} \wp(u).$$

$$X_1 \Leftrightarrow \mathcal{J}$$

$$(y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \Leftrightarrow \wp_u^2/4 = \wp^3 + \lambda_2 \wp^2 + \lambda_1 \wp + \lambda_0).$$

Remark The Jacobi inversion formula that $x = \wp(u)$, is non-trivial. By assuming it, due to the induced map $T_p \mathcal{J} \leftrightarrow T_p X_1$,

$$du = \frac{dx}{2y} \quad \text{implies that} \quad \frac{d}{du} = 2y \frac{d}{dx}.$$

Hence $\frac{d}{du} x = 2y \frac{d}{dx} x = 2y$.

Due to the Jacobi inversion formula, we have a corollary:

Corollary:

$$\text{“elliptic functions”} \equiv \left\{ \begin{array}{l} \text{rational functions of } x \text{ and } y \\ \text{whose arguments are } u = w(x, y) \end{array} \right\}$$

$$x(u) - 1, y(u) - 2, x^2(u) + ax(u) + b, \frac{1}{1+x^2(u)}, \frac{x}{y}(u), \frac{x^2(u)}{y(u)-a}, \dots$$

10. Differential relations

$$(a) \quad \wp_u^2 = 4(\wp^3 + \lambda_2 \wp^2 + \lambda_1 \wp + \lambda_0) \quad (y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0).$$

$$(b) \quad \wp_u \wp_{uu} = 6\wp^2 \wp_u + 4\lambda_2 \wp \wp_u + 2\lambda_1 \wp_u \text{ or}$$

$$\wp_{uu} - 6\wp^2 = 4\lambda_2 \wp + 2\lambda_1.$$

$$(c) \quad \wp_{uuu} = 12\wp \wp_u + 4\lambda_2 \wp_u \text{ (KdV equation } v_t + 6v v_x + v_{xxx} = 0); v = \wp/2.$$

11. Behaviour at ∞

$$y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \rightarrow y^2 \sim x^3 \text{ at } \infty.$$

Then the behaviour of x and y at ∞ is written by a local parameter t (6 is the least common multiple of 2 and 3).

$$x = \frac{1}{t^2}, \quad y = \frac{1}{t^3}(1 + o(t)), \quad \left(y^2 \sim x^3 \sim \frac{1}{t^6} \right)$$

On the other hand

$$du = \frac{dx}{2y} \sim \frac{-2/t^3 \cdot dt}{2/t^3(1 + o(t))} = (1 + o(t))dt$$

or

$$u = t(1 + o(t))$$

12. The commutative ring R , w and r

Let $R = \mathbb{C}[x, y]/(y^2 - x^3 - \dots)$ be identified with $\mathcal{O}_{X_1}(*\infty)$. We define

(a) the weight for an element ϕ of R :

$$\text{wt}(\phi) := \text{the order of singularity of } \phi \text{ at } \infty.$$

(b) The order of the vanishing of u at the origin of $f \in \mathcal{O}(\mathbb{C})$

$$r(f) := \text{order of zero of } f \text{ at the origin.}$$

13. Weierstrass Gap Table

Table 1		0	1	2	3	4	5	6	7	8	9	10
wt/r		1	-	x	y	x^2	xy	x^3	x^2y	x^4	x^3y	x^4y
$\langle u \rangle$		-	u	-	-	-	-	-	-	-	-	-

$$\phi_0 = 1, \phi_1 = x, \phi_2 = y, \phi_3 = x^2, \phi_4 = xy, \phi_5 = x^3, \phi_6 = x^2y, \dots$$

$$R = \bigoplus \mathbb{C}\phi_i \quad \text{as a vector space}$$

"-" = gap for ϕ .

In the gap, there is a semigroup $H := \{2^a 3^b\}$ generated by $(2, 3)$.

$$H = \{0, 2, 3, 4, 5, 6, \dots\}, \quad L = \mathbb{Z} \setminus H = \{1\}.$$

14. Theorem: Addition Formula: Frobenius-Stickelberger (1877) For $u^{(i)} := w(x_i, y_i)$,

$$\frac{\sigma(u^{(0)} + u^{(1)} + \dots + u^{(n)}) \prod_{i < j} \sigma(u^{(i)} - u^{(j)})}{\sigma^{n+1}(u^{(0)}) \sigma^{n+1}(u^{(1)}) \dots \sigma^{n+1}(u^{(n)})} = C_n \begin{vmatrix} 1 & x_0 & y_0 & x_0 y_0 & \dots & \phi_n(x_0, y_0) \\ 1 & x_1 & y_1 & x_1 y_1 & \dots & \phi_n(x_1, y_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & y_n & x_n y_n & \dots & \phi_n(x_n, y_n) \end{vmatrix}. \quad (1.1)$$

1) σ function of addition of multivariables $u^{(i)} = w(x_i, y_i)$ is given by this determinant.

2) $\phi_j(x_i, y_i)$ comes from the gap-table.

15. Motivation

(a) It should be recalled that when we apply the elliptic functions theory to the other fields, e.g., number theory, physics, engineer, information theory and so on, their explicit formulas and relations play important roles.

The elliptic function theory provides the algebraic geometrical properties and the analytic properties.

In order to apply the theory of functions of curves with higher genera to the other fields in future, we would like to have the similar theory and explicit relations and formulas of higher genus curves.

(b) By considering the integrable nonlinear equations, KdV equation, the Neumann system, the sine-Gordon equations, Mumford constructed such theories in "Tata Lectures on Theta II" for every hyperelliptic curves based upon Jacobi's style, which is known as Mumford triplet U, V, W .

We would like to proceed the further study of Tata Lecture Theta II on every affine plane curve based upon theories of Klein and Baker.

16. History of sigma functions

(a) Abel, Jacobi discovered holomorphic one forms over a hyperelliptic curve.

(b) Based upon the Jacobi sn, cn, dn functions, Weierstrass essentially discovered elliptic sigma functions 1841-2, which was called Al-function honour to Abel.

(c) Jacobi discovered the Abel theorem among curve and Jacobian. Jacobi found Jacobi inversion formulae (which is the base of Mumford's Tata II).

(d) 1856 Weierstrass discovered his theta function over every hyperelliptic curve, Al function, which was, later sophisticated to Klein's σ function by Klein as a hyperelliptic version of the Weierstrass σ function.

(e) 1857 Riemann discovered his Abelian function theory over every compact Riemann surface.

(f) 1856-80 Weierstrass constructed his elliptic function theory, which was known via his lecture.

(g) 1886 Klein defined the Kleinian σ function over every hyperelliptic curve.

(h) 1896 Baker wrote a book on the Kleinian σ function over every hyperelliptic curve and other one.

(i) 1903 Baker discovered the KdV hierarchy and KP equation over every hyperelliptic curve, which is known as the integrable nonlinear equations.

2 $g = 1$ curve revised: on sigma function

(Key idea of EEL-construction)

1.

$$\wp(u) = -\frac{d^2}{du^2} \log \sigma$$

means that we may find an entire function,

$$\sigma(u) \sim \exp \left(\int^u \int^{u'} \wp(u'') du' du'' \right).$$

We want to express the coordinate of the affine curve in terms of a generalized sigma function.

A meromorphic function h over \mathcal{J} whose zeros are $(a_i)_{i=1, \dots, N}$ and singularities are $(b_i)_{i=1, \dots, N}$ is given by

$$h(u; a, b) = h_0 e^{2(m\eta_1 + n\eta_2)u} \frac{\prod_{i=1}^N \sigma(u - a_i)}{\prod_{i=1}^N \sigma(u - b_i)}$$

where h_0 is a constant factor and $2m\omega_1 + 2n\omega_3 = \sum a_i - \sum b_i$.

2.

$$\wp(u-v) dudv \sim \frac{dudv}{(u-v)^2}, \quad \sigma(u-v) \sim (u-v), \quad \text{around } u \sim v$$

We may find the ratio of x 's and y 's to express $dudv/(u-v)^2$ and use the relation,

$$\int_{z_2}^{z_1} \int_{z_4}^{z_3} \frac{dw_1 dw_2}{2(w_1 - w_2)^2} = \int_{z_2}^{z_1} \left(\frac{dw_1}{(w_1 - z_3)} - \frac{dw_1}{(w_1 - z_4)} \right) = \log \left(\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \right)$$

to characterise the sigma function.

3. For the curve $y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$, and $P_i = (x_i, y_i)$

$$u_i = \int_{\infty}^{(x_i, y_i)} \nu^I,$$

$$\begin{aligned} \Omega(P_1, P_2) &:= \wp(u_1 - u_2) \nu^I(P_1) \nu^I(P_2) \\ &= \frac{F(P_1, P_2) dx_1 \otimes dx_2}{(x_1 - x_2)^2 4y_{P_1} y_{P_2}}, \end{aligned}$$

where F is an element of $R \otimes R$,

$$F(P_1, P_2) := f(P_1, P_2) + y_1 y_2,$$

$$\begin{aligned} f(P_1, P_2) &:= \sum_{j=0}^g x_1^j x_2^j (\lambda_{2j+1}(x_1 + x_2) + 2\lambda_{2j}) \\ &= x_1 x_2 (x_1 + x_2 + 2\lambda_2) + \lambda_1 (x_1 + x_2) + 2\lambda_0. \end{aligned}$$

4. **Fundamental differential of the 2nd kind:** We call the two-form $\Omega(P_1, P_2)$ on $X \times X$ **FDII**

- (a) Symmetric: $\Omega(P_1, P_2) = \Omega(P_2, P_1)$
- (b) only pole (of 2nd order) along the diagonal of $X \times X$, and in the vicinity of each point (P_1, P_2) is expanded in power series as

$$\Omega(P_1, P_2) = \left(\frac{1}{(t_{P_1} - t_{P_2})^2} + d_{\geq 1} \right) dt_{P_1} \otimes dt_{P_2} \quad (\text{as } P_1 \rightarrow P_2),$$

- (c) $\Omega(P_1, P_2)$ is written only by the affine coordinate.

5. **The Riemann fundamental relation:** Due to the idea,

$$\int_{z_2}^{z_1} \int_{z_4}^{z_3} \frac{dw_1 dw_2}{2(w_1 - w_2)^2} = \log \left(\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \right),$$

we have the relation.

Theorem (Riemann fundamental relation): For $(P, Q, P', Q') \in X^4$,

$$u := w(P), \quad u' := w(P'), \quad v := w(Q), \quad v' := w(Q'),$$

$$\exp \left(\int_Q^P \int_{Q'}^{P'} \Omega(P_1, P_2) \right) = \frac{\sigma(u - u')\sigma(v - v')}{\sigma(u - v')\sigma(v - u')}.$$

Remark From this relation, we have the Jacobi inversion formulae. For example, applying $\partial/\partial u$ and $\partial/\partial u'$ it, it recovers the relation $\wp(u_1 - u_2)$.

6. To determine the FDII, we find a way (**Klein, Bolza, Baker, Eilbeck-Enolskii-Leykin construction [EEL]**)

Let $\Sigma(P, Q)$ be the following form,

$$\Sigma(P, Q) := \frac{y_P + y_Q}{(x_P - x_Q)2y_P} dx_P. \quad (2.1)$$

Noting $\Sigma(P, Q)$ has the properties

- (a) $\Sigma(P, Q)$ as function of P has singular at $Q = (x_Q, y_Q)$, vanishes at $\iota_H(Q) = (x_Q, -y_Q)$ and singular at ∞ .
- (b) $\Sigma(P, Q)$ as function of Q has singular at P and at ∞ .

7. **Lemma:** ($\Omega(P_1, P_2)$ and $\Sigma(P_1, P_2)$)

$$\Omega(P_1, P_2) = d_{P_2} \Sigma(P_1, P_2) + \nu^I(P_1) \otimes \nu^{II}(P_2)$$

where

$$d_Q \Sigma(P, Q) := dx_P \otimes dx_Q \frac{\partial}{\partial x_Q} \frac{y_P + y_Q}{(x_P - x_Q)2y_P} dx_P. \quad (2.2)$$

By the differentials of 2nd kind ν^{II} whose pole is at ∞ , we remove the singularity of $d_Q \Sigma(P, Q)$ at ∞ .

8. **Lemma:** ($\Sigma(P_1, P_2)$ and differentials of 2nd kind ν^{II})

$$\begin{aligned} d_Q \Sigma(P, Q) - d_P \Sigma(Q, P) &= \nu^I(Q) \otimes \nu^{II}(P) - \nu^I(P) \otimes \nu^{II}(Q) \\ &= \frac{dx_Q}{2y_Q} \otimes \frac{x_P dx_P}{2y_P} - \frac{dx_P}{2y_P} \otimes \frac{x_Q dx_Q}{2y_Q} \end{aligned} \quad (2.3)$$

This leads to the Legendre relation. We should note that this idea appeared in [Ga] for a curve.

3 $g = 2$ Curve (hyperelliptic curve)

Let us consider the hyperelliptic curve:

$$X_2 := \{(x, y) \mid y^2 = x^5 + \lambda_4 x^4 + \cdots + \lambda_0\} \cup \infty,$$

and let $\lambda_5 = 1$.

1. Abelian integral: $w : X_2 \rightarrow \mathbb{C}^2; \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = w(P) \right), P = (x, y)$

$$w(P) := \left(\int_{\infty}^P \frac{dx}{2y}, \int_{\infty}^P \frac{x dx}{2y} \right)^t, \quad \nu^I_1 := \frac{dx}{2y}, \quad \nu^I_2 := \frac{x dx}{2y}.$$

ν^I_a is the holomorphic one-form, differential of the 1st kind.

2. We have a correspondence

$$w : S^2 X_2 \rightarrow \mathbb{C}^2;$$

$$w(P_1, P_2) := w(P_1) + w(P_2).$$

$$\text{Path}(S^2(X_2)) \rightarrow \mathbb{C}^2 \quad (\text{surjection}).$$

3. The w naturally defines the periodic (2×4) matrix ω and the lattice Λ :

$$\omega := (\omega' \quad \omega''), \quad \Lambda := 2w \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \\ \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} \subset \mathbb{C}^2,$$

$$\omega' := \frac{1}{2} \begin{pmatrix} \int_{\alpha_1} \nu^I_1 & \int_{\alpha_2} \nu^I_1 \\ \int_{\alpha_1} \nu^I_2 & \int_{\alpha_2} \nu^I_2 \end{pmatrix}, \quad \omega'' := \frac{1}{2} \begin{pmatrix} \int_{\beta_1} \nu^{II}_1 & \int_{\beta_2} \nu^{II}_1 \\ \int_{\beta_1} \nu^{II}_2 & \int_{\beta_2} \nu^{II}_2 \end{pmatrix}.$$

4. $\kappa : \mathbb{C}^2 \rightarrow \mathcal{J} := \mathbb{C}^2/\Lambda$. \mathcal{J} is the Jacobian.

5. The Abel theorem is that $\kappa \circ w : S^2 X_2 \rightarrow \mathcal{J}$ is birational.

6. **Realization of $\Sigma(P_1, P_2)$:**

Let $\Sigma(P, Q)$ be the following form,

$$\Sigma(P, Q) := \frac{y_P + y_Q}{(x_P - x_Q)2y_P} dx_P. \quad (3.1)$$

7. Noting $\Sigma(P, Q)$ has the properties:

- (a) $\Sigma(P, Q)$ as function of P has singular at Q , vanishes at $\iota_H(Q)$ and singular at ∞ .
- (b) $\Sigma(P, Q)$ as function of Q has singular at P and at ∞ .

8. **Realization of differentials of the 2nd kind ν^{II} :** (Weierstrass, Clebsh, Bolza [B1, B2]) The differentials of 2nd kind ν^{II}_j ($j = 1, 2$) whose pole is at ∞ ;

$$\begin{aligned} d_Q \Sigma(P, Q) - d_P \Sigma(Q, P) \\ = \sum_{i=1}^2 \left(\nu^I_i(Q) \otimes \nu^{II}_i(P) - \nu^I_i(P) \otimes \nu^{II}_i(Q) \right) \end{aligned} \quad (3.2)$$

Then the differential of the 2nd kind is determined as

$$\nu^{II}_1 = \frac{3x^3 + 2\lambda_4 + 3\lambda_3}{2y}, \quad \nu^{II}_2 = \frac{x^2}{2y},$$

up to holomorphic one-forms.

9. **The fundamental differential of the 2nd kind $\Omega(P_1, P_2)$** in terms of the coordinate of hyperelliptic curve

$$\begin{aligned} \Omega(P_1, P_2) &= d_{P_2} \Sigma(P_1, P_2) + \sum_{i=1}^g \nu^I_i(P_1) \otimes \nu^{II}_i(P_2) \\ &= \frac{F(P_1, P_2) dx_1 \otimes dx_2}{(x_1 - x_2)^2 4y_{P_1} y_{P_2}}, \end{aligned} \quad (3.3)$$

where F is an element of $R \otimes R$,

$$F(P_1, P_2) = f(P_1, P_2) + y_1 y_2,$$

$$f(P_1, P_2) = \sum_{j=0}^g x_1^j x_2^j (\lambda_{2j+1}(x_1 + x_2) + 2\lambda_{2j}).$$

10. **Proposition (Legendre relation) [B1]:** For the complete integral of the 2nd kind is given by

$$\eta' := \frac{1}{2} \begin{pmatrix} \int_{\alpha_1} \nu^{II}_1 & \int_{\alpha_2} \nu^{II}_1 \\ \int_{\alpha_1} \nu^{II}_2 & \int_{\alpha_2} \nu^{II}_2 \end{pmatrix}, \quad \eta'' := \frac{1}{2} \begin{pmatrix} \int_{\beta_1} \nu^{II}_1 & \int_{\beta_2} \nu^{II}_1 \\ \int_{\beta_1} \nu^{II}_2 & \int_{\beta_2} \nu^{II}_2 \end{pmatrix}.$$

and the Legendre relations is given by

$$\eta' \omega''' - \eta'' \omega'^t = \frac{\pi}{2} I_{2 \times 2}.$$

11. **Definition (Kleinian σ function) :** Klein defined Kleinian σ function as an extension of the Weierstrass σ function (1886,8), which is a nice θ function is defined by

$$\sigma(u) := \gamma_0 e^{-u^t \omega'^{-1} \eta'^t u} \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (\omega'^{-1} u; \omega'^{-1} \omega''),$$

where γ_0 is constant, δ 's are theta characteristics,

$$\delta' = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \quad \delta'' = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$

and the Riemann theta function is given by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z; \tau) = \sum_{n \in \mathbb{Z}^2} e^{(\pi \sqrt{-1}((n+a)^t \tau (n+a) - (n+a)^t (z+b)))}.$$

12. **Proposition: properties of σ :**

(a) **translational formulae:** For $u, v \in \mathbb{C}^g$, and $\ell \in \mathbb{Z}^4$ ($= 2\omega' \ell' + 2\omega'' \ell''$) $\in \Lambda$, we define

$$L(u, v) := 2^t u (\eta' v' + \eta'' v''),$$

$$\chi(\ell) := \exp[\pi \sqrt{-1} (2({}^t \ell' \delta'' - {}^t \ell'' \delta') + {}^t \ell' \ell'')].$$

The following holds

$$\sigma(u + \ell) = \sigma(u) \exp(L(u + \frac{1}{2}\ell, \ell)) \chi(\ell). \quad (3.4)$$

(b) **zeros of σ :**

$$\{\text{zeros of } \sigma\} = \kappa^{-1} \Theta,$$

where the θ divisor Θ is given by $\kappa \circ w : X_2 \rightarrow \Theta \subset \mathcal{J}$.

13. **Theorem (The Riemann fundamental relation):** For $(P, Q, P_i, P'_i) \in X^2 \times (S^2(X) \setminus S_1^2(X)) \times (S^2(X) \setminus S_1^2(X))$,

$$u := \sum_{i=1}^2 w(P_i), \quad v := \sum_{i=1}^2 w(P'_i),$$

$$\exp \left(\sum_{i,j=1}^2 \Omega_{P_i, P'_j}^{P, Q} \right) = \frac{\sigma(w(P) - u) \sigma(w(Q) - v)}{\sigma(w(Q) - u) \sigma(w(P) - v)}$$

$$= \frac{\sigma(w(P) - w(P_1, \dots, P_4)) \sigma(w(Q) - w(P'_1, \dots, P'_4))}{\sigma(w(Q) - w(P_1, \dots, P_4)) \sigma(w(P) - w(P'_1, \dots, P'_4))}$$

14. **Definition: (Kleinian \wp functions and ζ functions)**

(a) The Kleinian \wp functions

$$\wp_{ij} := -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma \in \Gamma(\mathcal{J}, \mathcal{O}(2\Theta)).$$

(b) The Kleinian ζ functions

$$\zeta_i := \frac{\partial}{\partial u_i} \log \sigma.$$

15. **Theorem: Jacobi inversion formula** (Jacobi 1846 [Ja]) For $((x_1, y_1), (x_2, y_2)) \in S^2 X_2$ and $u := w((x_1, y_1), (x_2, y_2)) \in \mathbb{C}^2$,

$$x^2 + \wp_{22}(u)x + \wp_{21}(u) = x^2 - (x_1 + x_2)x + x_1 x_2$$

$$= \frac{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x & x^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}}.$$

This implies

$$\mathcal{J} \equiv \kappa w(S^2 X_2) \rightarrow S^2 X_2,$$

16.

“hyperelliptic functions” $\equiv \left\{ \begin{array}{l} \text{rational functions of} \\ \text{symmetric functions} \\ \text{of } x_1, y_1, x_2, y_2 \text{ whose arguments} \\ \text{are } u = w((x_1, y_1), (x_2, y_2)) \end{array} \right\}$

$$(x_1 x_2)(u), (x_1 + x_2)(u), (y_1 + y_2)(u), \frac{x_1 x_2}{y_1 + y_2}(u), \dots$$

17. **Theorem: Jacobi inversion formula over a strata** (Grant 1988)

The stratification of Jacobian:

$$\{0\} \subset W^1 \subset \mathcal{J}, \quad W^1 := \kappa w(X_2), \quad \mathcal{J} \equiv \kappa w(S^2 X_2).$$

For $u = w(x_1, y_1) \in \mathbb{C}^2$,

$$x + \frac{\sigma_1}{\sigma_2}(u) = \frac{\begin{vmatrix} 1 & x_1 \\ 1 & x \end{vmatrix}}{\begin{vmatrix} 1 \\ 1 \end{vmatrix}} = x - x_1, \quad \text{i.e., } x_1 = \frac{\sigma_1}{\sigma_2}(u),$$

where $\sigma_i := \partial \sigma / \partial u_i$.

$$W^1 \equiv \kappa w(X_2) \equiv X_2,$$

18. **Theorem: Differential identical relations: KdV hierarchy (Baker 1903) [B3]**

$$\wp_{2222} - 6\wp_{22}^2 = \frac{1}{8} \lambda_3 + \lambda_4 \wp_{22} + \wp_{21}$$

$$\wp_{2221} - 6\wp_{22} \wp_{21} = \lambda_4 \wp_{21} - \frac{1}{2} \wp_{11}$$

$$\wp_{2211} - 4\wp_{21}^2 - 2\wp_{22} \wp_{11} = \frac{1}{2} \lambda_3 \wp_{21}$$

$$\wp_{2111} - 6\wp_{22} \wp_{12} = -\frac{1}{4} \lambda_0 - \frac{1}{2} \lambda_1 \wp_{22} + \lambda_2 \wp_{21}$$

$$\wp_{1111} - 6\wp_{11} = -\frac{1}{4} \lambda_0 \lambda_4 + \frac{1}{8} \lambda_1 \lambda_3$$

$$-3\lambda_0 \wp_{22} + \lambda_1 \wp_{21} + \lambda_2 \wp_{11}$$

(KdV equation: $v_t + 6v v_x + v_{xxx} = 0$,

$$t = v_1 - \lambda_4 v_2, \quad x = v_2, \quad v = \wp_{22}/2).$$

19. **Theorem: differential identical relation II:**

dispersionless KdV equation [MP09b] For $u = w(x, y) \in \mathbb{C}^2$,

$$\frac{\partial}{\partial u_1} x = x \frac{\partial}{\partial u_2} x.$$

proof: due to the relation $2y \frac{d}{dx} x = x \frac{2y}{x} \frac{d}{dx} x$. ($du_1 = \frac{dx}{2y}$, $du_2 = \frac{x dx}{2y}$.)

KdV equation: $u_t + 6u u_x + u_{xxx} = 0$.

dKdV equation: $u_t + 6u u_x = 0$.

20. Behaviour at ∞

$$y^2 = x^5 + \lambda_4 x^4 + \dots + \lambda_1 x + \lambda_0 \rightarrow y^2 \sim x^5 \text{ at } (\infty, \infty) \equiv \infty.$$

Then the behaviour of x and y at ∞ is written by a local parameter t (10 is the least common multiple of 2 and 5).

$$x = \frac{1}{t^2}, \quad y = \frac{1}{t^5}(1 + o(t)), \quad \left(y^2 \sim x^5 \sim \frac{1}{t^{10}} \right)$$

On the other hand $du_1 = \frac{dx}{2y}$ and $du_2 = \frac{xdx}{2y}$,

$$du_1 \sim \frac{-2/t^3 \cdot dt}{2/t^5(1+o(t))} = t^2(1+o(t))dt,$$

$$du_2 \sim \frac{-2/t^3 \cdot 1/t^2 \cdot dt}{2/t^5(1+o(t))} = (1+o(t))dt$$

or

$$u_1 = t^3(1+o(t)), \quad u_2 = t(1+o(t))$$

21. Weierstrass Gap Table

The distribution of wt and r of $y^2 = x^5 + \dots$ are given by the Table 2:

wt/r	0	1	2	3	4	5	6	7	8	9	10
ϕ	1	-	x	-	x^2	y	x^3	xy	x^4	x^2y	x^5
(u_1, u_2)	-	u_2	-	u_1	-	-	-	-	-	-	-
		3	2	1	0						

$\phi_0 = 1, \phi_1 = x, \phi_2 = x^2, \phi_3 = y, \phi_4 = x^2, \phi_5 = xy, \phi_6 = x^3, \dots$

$$R = \mathbb{C}[x, y]/(y^2 - x^5 - \dots) = \bigoplus \mathbb{C}\phi_i \quad \text{as a vector sp.}$$

“-” = gap for ϕ .

In the gap, there is a semigroup $H := \{2^a 5^b\}$ generated by (2, 5).

$$H = \{0, 2, 4, 5, 6, \dots\}, \quad L = \mathbb{Z} \setminus H = \{1, 3\}.$$

22. Theorem: Addition Formula; Frobenius-Stickelberger Relation (Ônishi 1997 [O1]) For $u^{(i)} := w(x_i, y_i)$,

$$\frac{\sigma(u^{(0)} + u^{(1)} + \dots + u^{(n)}) \prod_{i < j} \sigma(u^{(i)} - u^{(j)})}{\sigma^{n+1}(u^{(0)}) \sigma^{n+1}(u^{(1)}) \dots \sigma^{n+1}(u^{(n)})} = c_n \begin{vmatrix} 1 & x_0 & x_0^2 & y_0 & \dots & \phi_n(x_0, y_0) \\ 1 & x_1 & x_1^2 & y_1 & \dots & \phi_n(x_1, y_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & y_n & \dots & \phi_n(x_n, y_n) \end{vmatrix} \quad (3.5)$$

1) σ function of addition of multivariables $u^{(i)} = w(x_i, y_i)$ is given by this determinant.

2) $\phi_j(x_i, y_i)$ comes from the gap-table.

23. Summary: $g = 2$ hyperelliptic case

- (a) The formulas are simple, explicit and concrete like ones of elliptic functions.
- (b) Even though σ function is a transcendent with respect to u , its behaviours are algebraic, e.g., it generates meromorphic functions of coordinates of the plane curves.

24. Remark: The generalisation to every hyperelliptic curve (Klein 1886,8, Bolza 1895, Baker 1903, Buchsterber-Enolskii-Leykin, 1997, Ônishi (Iwate) 2005, Enolski-Gibbons 2002, Previato-M 2008) [B1, B2, B3, BEL, O1]

(a) Hyperelliptic curve is given by

$$y^2 = x^{2g+1} + \lambda_{2g} x^{2g} + \dots + \lambda_1 x + \lambda_0.$$

(b) Theorem: Jacobi inversion formula (Klein 1886,8, Baker 1897)

$$\wp_{ij} = \frac{\partial^2}{\partial u_i \partial u_j} \log \sigma.$$

$\mathcal{J} = \kappa w(S^g X_g)$ case: $u = w((x_i, y_i)_{i=1, \dots, g})$.

$$x^g + \sum_{i=0}^{g-1} \wp_{gg-i}(u) x^i = \frac{\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^g \\ 1 & x_2 & x_2^2 & \dots & x_2^g \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_g & x_g^2 & \dots & x_g^g \\ 1 & x & x^2 & \dots & x^g \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{g-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{g-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_g & x_g^2 & \dots & x_g^{g-1} \end{vmatrix}}$$

\wp_{ij} is elementary symmetric function of x_1, \dots, x_g .

(c) Theorem: Jacobi inversion formulae (Enolskii 2002(unpublished), MP 2008) Let the stratification of Jacobian be

$$\{0\} \subset W^1 \subset W^2 \subset \dots \subset \mathcal{J} \equiv W^g, \quad W^k := \kappa w(S^k X_2).$$

For $W^k := \kappa w(S^k X_g)$ case ($k < g$) and $u = w((x_i, y_i)_{i=1, \dots, k})$ we have

$$x^k + \sum_{i=0}^{k-1} \frac{\sigma_{k-i+1}(u) x^i}{\sigma_{k+1}(u)} = \frac{\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & x_k^2 & \dots & x_k^k \\ 1 & x & x^2 & \dots & x^k \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & x_k^2 & \dots & x_k^{k-1} \end{vmatrix}}, \quad \frac{\sigma_1(u)}{\sigma_2(u)} = x_1,$$

where $\sigma_i := \partial \sigma / \partial u_i$.

(d) Theorem: Addition Formula (EEMOP 07) For $u := \sum^m w(x_i, y_i)$, and $v := \sum^n w(x'_i, y'_i)$ ($m, n \geq g$),

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)} = \delta(g, m, n) \times \frac{\prod_{i=0}^1 \Delta_{m+n}((x_1, y_1), \dots, (x_m, y_m), (x'_1, (-1)^i y'_1), \dots, (x'_n, (-1)^i y'_n))}{(\Delta_m((x_1, y_1), \dots, (x_m, y_m)) \Delta_n((x'_1, y'_1), \dots, (x'_n, y'_n)))^2} \times \prod_{i=1}^m \prod_{j=1}^n \frac{1}{\Delta_2((x_i, y_i), (x'_j, y'_j))}$$

where $\delta(g, m, n) = (-1)^{gn + \frac{1}{2}n(n-1) + mn}$.

$$\Delta_n((x_1, y_1), \dots, (x_n, y_n)) = \begin{vmatrix} 1 & \phi_1(x_1, y_1) & \phi_2(x_1, y_1) & \dots & \phi_{n-1}(x_1, y_1) \\ 1 & \phi_1(x_2, y_2) & \phi_2(x_2, y_2) & \dots & \phi_{n-1}(x_2, y_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(x_{n-1}, y_{n-1}) & \phi_2(x_{n-1}, y_{n-1}) & \dots & \phi_{n-1}(x_{n-1}, y_{n-1}) \\ 1 & \phi_1(x_n, y_n) & \phi_2(x_n, y_n) & \dots & \phi_{n-1}(x_n, y_n) \end{vmatrix}$$

(e) Summary : every hyperelliptic curve The following properties preserves even for every hyperelliptic curve.

- i. The formulas are simple, explicit and concrete like ones of elliptic functions.
- ii. Even though σ function is a transcendent with respect to u , its behaviours are algebraic, e.g., it generates meromorphic functions of coordinates of the plane curves.

4 $g = 3$ Curve (non-hyperelliptic curve)

We will consider a non-hyperelliptic genus $g = 3$ curve

$$X_3 := \{(x, y) \mid y^3 = x^4 + \lambda_3 x^3 + \dots + \lambda_0\} \cup \infty.$$

1. Abelian integral: $w : \text{Path}(X_3) \rightarrow \mathbb{C}^3$, $w(P) := \int_{\infty}^P \begin{pmatrix} \nu^I_1 \\ \nu^I_2 \\ \nu^I_3 \end{pmatrix}$;

$$\nu^I_1 := \frac{dx}{3y^2}, \quad \nu^I_2 := \frac{xdx}{3y^2}, \quad \nu^I_3 := \frac{dx}{3y}.$$

2. Extended to its domain to $S^a(X_3)$ ($a = 1, 2, 3$) by

$$w : \text{Path}(S^a X_3) \rightarrow \mathbb{C}^3; \quad w(P_1, \dots, P_a) = \sum_{i=1}^a w(P_i).$$

3. The w naturally define the periodic matrix and the lattice $\Lambda \in \mathbb{C}^3$.

$$\kappa\mathbb{C}^3 \rightarrow \mathcal{J} := \mathbb{C}^3/\Lambda$$

4. The Abel theorem is that $\kappa \circ w : S^3 X_3 \rightarrow \mathcal{J}$ is birational.

5. Buchsterber-Enolskii-Leykin defined Kleinian σ function over X_3 as an extension of the Weierstrass σ function (BEL 1999, 2000, EEL2000).

6. The Kleinian \wp functions

$$\wp_{ij} := \frac{\partial}{\partial u_i \partial u_j} \log \sigma.$$

7. Eilbeck-Enolskii-Leykin (EEL) construction:

$$\Sigma(P, Q) := \frac{y_P^2 + y_P y_Q + y_Q^2}{(x_P - x_Q) 3y_P^2} dx_P. \quad (4.1)$$

Noting $\Sigma(P, Q)$ has the properties:

- (a) $\Sigma(P, Q)$ as function of P has singular at Q , vanishes at $\iota_H(Q)$ and singular at ∞ .
- (b) $\Sigma(P, Q)$ as function of Q has singular at P and at ∞ .

8. The differentials of the 2nd kind:

$$\nu^{II}_1 = \frac{-(5x^2 - 3\lambda_3 x + \lambda_6) y dx}{3y^2}, \quad \nu^{II}_2 = \frac{-2xy dx}{3y^2},$$

$$\nu^{II}_3 = \frac{-x^2 dx}{3y^2}.$$

9. **Weierstrass Gap Table:** For $X_3 : y^3 = x^4 + \lambda_3 x^3 + \dots + \lambda_0$ we have the gap table:

Table 3											
wt/r	0	1	2	3	4	5	6	7	8	9	10
ϕ	1	-	-	x	y	-	x^2	xy	y^2	x^3	$x^2 y$
$\langle u \rangle$	-	u_3	u_2	-	-	u_1	-	-	-	-	-
	5	4	3	2	1	0					

$\phi_0 = 1, \phi_1 = x, \phi_2 = y, \phi_3 = x^2, \phi_4 = xy, \dots$ In the gap, there is a semigroup $H := \{3^a 4^b\}$ generated by $(3, 4)$.

$$H = \{0, 3, 4, 6, \dots\}, \quad L = \mathbb{Z} \setminus H = \{1, 2, 5\}.$$

10. **Jacobi inversion formula** (Enolskii Eilbeck Leykin 2000, Buchsterber Enolskii Leykin 1999)

Theorem: For $u = w((x_1, y_1), (x_2, y_2), (x_3, y_3))$,

$$x^2 + \wp_{33}(u)y + \wp_{32}(u)x + \wp_{31}(u) = \frac{\begin{vmatrix} 1 & x_1 & y_1 & x_1^2 \\ 1 & x_2 & y_2 & x_2^2 \\ 1 & x_3 & y_3 & x_3^2 \\ 1 & x & y_1 & x^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}$$

$$\mathcal{J} = S^3 X_3$$

11. **Jacobi inversion formulae** [MP08]

Theorem: Let us consider the stratification of Jacobian:

$$\{0\} \subset W^1 \subset W^2 \subset \mathcal{J} \equiv W^3, \quad W^k := \kappa w(S^k X_3).$$

For $u = w((x_1, y_1), (x_2, y_2))$,

$$y + \frac{\sigma_2}{\sigma_3}(u)x + \frac{\sigma_1}{\sigma_3}(u) = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}},$$

where $\sigma_i := \partial \sigma / \partial u_i$.

For $u = w((x_1, y_1))$,

$$x + \frac{\sigma_1}{\sigma_2}(u) = \frac{\begin{vmatrix} 1 & x_1 \\ 1 & x \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x \end{vmatrix}}.$$

12. **Differential relations** [EEL, EEMOP08]

$$\begin{aligned} \wp_{3333} &= 6\wp_{33}^2 + \wp_{22} \\ \wp_{2333} &= 6\wp_{32}\wp_{33} + 3\lambda_3\wp_{33} \\ \wp_{2233} &= 6\wp_{32}^2 + 2\wp_{33}\wp_{22} + 2\lambda_2 + 4\wp_{13} \\ \wp_{2223} &= 6\wp_{22}\wp_{23} + 3\lambda_3\wp_{22} \\ \wp_{2222} &= 6\wp_{22}^2 + 24\wp_{13}\wp_{33} - 3\lambda_3^2\wp_{33} + 12\lambda_2\wp_{33} \\ &\dots\dots\dots \\ &\dots\dots\dots \\ &\dots\dots\dots \end{aligned}$$

13. **Differential relations II** [MP09]

(dKdV equation:) For $u \in w(X_2) \subset \mathbb{C}^2$,

$$\frac{\partial}{\partial u_1} x = x \frac{\partial}{\partial u_3} x.$$

it is proved by the fact $2y \frac{d}{dx} x = x \frac{2y}{x} \frac{d}{dx} x$.

5 $y^r = f(x)$ Curve (C_{rs} curve)

[BLE1, BLE2, BLE3, EEL, MP08, MP09a]

1. Let us consider a more general plane curve of cyclic (r, s) type

$$X^g := \{(x, y) \mid y^r = x^s + \dots + \lambda_0\} \cup \infty.$$

where r and s are positive integers of $r < s$ and $(r, s) = 1$.

2. Let $R := \mathbb{C}[x, y]/(y^r - x^s - \dots - \lambda_0)$ and ϕ_i be a monomial in R such that $wt(\phi_{i-1}) < wt(\phi_i)$, $\phi_0 = 1$ and

$$R = \bigoplus_{i=0}^{\infty} \mathbb{C}\phi_i.$$

3. **Jacobi inversion formula** $X^g : y^r = x^s + \dots + \lambda_0$:

Theorem: Jacobi inversion formulae [MP08] $\mathcal{J} = \kappa w(S^g X_g)$ case: $u = w((x_1, y_1), \dots, (x_g, y_g)) \in \mathbb{C}^g$

$$\phi_g(P) + \sum_{i=0}^{g-1} \wp_{gg-i}(u)\phi_i(P) = \frac{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \dots & \phi_g(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \dots & \phi_g(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_g) & \phi_2(P_g) & \dots & \phi_g(P_g) \\ 1 & \phi_1(P) & \phi_2(P) & \dots & \phi_g(P) \end{vmatrix}}{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \dots & \phi_{g-1}(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \dots & \phi_{g-1}(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_g) & \phi_2(P_g) & \dots & \phi_{g-1}(P_g) \end{vmatrix}}$$

4. Jacobi inversion formulae of strata of Jacobian

Theorem: Jacobi inversion formulae [MP08] The stratification of Jacobian:

$$\{0\} \subset W^1 \subset W^2 \subset \dots \subset \mathcal{J} \equiv W^g, \quad W^k := \kappa w(S^k X_2).$$

$W^k := \kappa w(S^k X_g)$ case ($i < g$)

$$\phi_k(P) + \sum_{i=0}^{k-1} \frac{\sigma_{k-i+1}}{\sigma_{k+1}} x^i \phi_i(P) = \frac{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \dots & \phi_k(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \dots & \phi_k(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_g) & \phi_2(P_g) & \dots & \phi_k(P_g) \\ 1 & \phi_1(P) & \phi_2(P) & \dots & \phi_k(P) \end{vmatrix}}{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \dots & \phi_{k-1}(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \dots & \phi_{k-1}(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_g) & \phi_2(P_g) & \dots & \phi_{k-1}(P_g) \end{vmatrix}}$$

5. Summary : (r, s) curve

The following properties preserves even for every curve of $y^r = x^s + \lambda_{s-1}x^{s-1} + \dots$.

- (a) The formulas are simple, explicit and concrete like ones of elliptic functions.
- (b) Even though σ function is a transcendent with respect to u , its behaviours are algebraic, e.g., it generates meromorphic functions of coordinates of the plane curves.

6 Projects and Problems

1. Revise the Riemann-Kempf singularity theorem in terms of this theoretical framework. (Almost done!) (M-Previato 2009)
2. Define the sigma function over a space curve, extend the Jacobi inversion formula (Almost done!) (Komeda-M-Previato 2009), and investigate their properties.
3. As the sigma function is modular invariant, use its properties and consider the moduli space (partially started by Buchstaber and Leykin 2003)!
4. As we have the Frobenius-Stickelberger for the cyclic curves, extend it to more general curve with Galois action.
5. As the structure of the addition formula of hyperelliptic curve is revealed well, write it down for general curves.
6. Extend every relation in the elliptic curve to one for more general curve.
7. Apply them to physical and number theoretic problems. [vGP, MP09b, O1, Ma1,2]

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