Sigma functions and Space Curve

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Abstract

In this talk, I show that Kleinian sigma function, which is a generalization of Weierstrass elliptic sigma function, is extended to space curves, (3,4,5), (3,7,8) and (6,13,14,15,16) type. In terms of the function, the Jacobi inversion formula is also generalized, in which the affine coordinates are given as functions of strata of Jacobi variety associated with these curves.

Joint with Emma Previato, Jiryo Komeda

1 Notations

Cyclic $(3,4,5)$ **curve**
Commutative Rings: $R := R_2$, R_3 and R_4

1) $R_2 := \mathbb{C}[x, y_4, y_5]/(f_a = 0, a = 4, 5, 9)$
$f_9 := y_4 y_5 - k_2(x)k_1(x)$, $f_5 := y_5 - \frac{y_4^2}{k_1(x)}$,
$f_4 := y_4 - \frac{y_5^2}{k_2(x)}$
where $k_2(x) := (x - b_1)(x - b_2), = x^2 + \lambda_1^{(2)} x + \lambda_2^{(2)}$,
$k_1(x) := (x - b_0)$
2) $R_3 := \mathbb{C}[x, y_4]/(y_4^3 - k_4(x))$,
3) $R_4 := \mathbb{C}[x, y_5]/(y_5^3 - k_5(x))$,

$$k_4(x) := k_2(x)k_1(x)^2, \quad k_5(x) := k_2(x)^2 k_1(x),$$

Hom. of these rings: $\xi_4 : R_4 \to R, \quad \xi_5 : R_5 \to R$
Algebraic Curve: X :its affine part Spec R $R = \mathcal{O}_X(*\infty)$
the sheaf of holomorphic functions over X**:** $\mathcal{O}_X$:
the Jacobian of X**:** $\mathcal{J}$
w-degree: $\deg_w : R \to \mathbb{Z}$, (Order of the singularity at ∞)
$\deg_w(x) = 3$, $\deg_w(y_4) = 4$, $\deg_w(y_5) = 5$:
The local parameter t_∞ at ∞,

$$x = \frac{1}{t_\infty^3},\ y_a = \frac{1}{t_\infty^a}\left(1 + d_>(t_\infty)\right),$$

Behavior at ∞**:** (Table 1)

Table 1

	0	1	2	3	4	5	6	7	8	9	10	11	12
X_3	1	-	-	x	y_4	-	x^2	xy_4	y_4^2	x^3	x^2y_4	xy_4^2	x^4
X_4	1	-	-	x	-	y_5	x^2	-	xy_5	x^3	y_5^2	x^2y_5	x^4
X_2	1	-	-	x	y_4	y_5	x^2	xy_4	xy_5	y_4y_5	x^2y_4	x^2y_5	xy_4y_5

the (monic) monomial for a non-negative integer n $\phi_n \in R$:

$$R_a = \oplus_{n=0}\mathbb{C}\phi_n^{(a)},\ \text{as a vector space}$$

$\phi_0^{(2)} = 1$, $\phi_1^{(2)} = x$, $\phi_2^{(2)} = y_4$, $\phi_3^{(2)} = y_5$, $\phi_4^{(2)} = x^2$, $\cdots$
$\phi_0^{(3)} = 1$, $\phi_1^{(3)} = x$, $\phi_2^{(3)} = y_4$, $\phi_3^{(3)} = x^2$, $\phi_4^{(3)} = xy_4$, $\cdots$.
$\phi_0^{(4)} = 1$, $\phi_1^{(4)} = x$, $\phi_2^{(4)} = y_5$, $\phi_3^{(4)} = x^2$, $\phi_4^{(4)} = xy_5$, $\cdots$.

the order of pole at ∞ $N^{(a)}(n) := \deg_w(\phi_n^{(a)})$ such that $N^{(a)}(n) < N^{(a)}(n+1)$.

$$\phi_n^{(a)}(P) = \frac{1}{t_\infty^{N^{(a)}(n)}}\left(1 + d_>(t_\infty)\right).$$

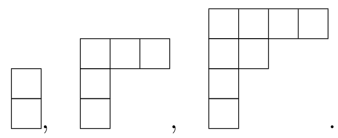
Numerical Semigroup $N^{(a)}(n)$ is a numerical semigroup.

Table 2

	0	1	2	3	4	5	6	7	8	9	10	11	12
(3, 4)	●	-	-	●	●	-	●	●	●	●	●	●	●
(3, 5)	●	-	-	●	-	●	●	-	●	●	●	●	●
(3, 4, 5)	●	-	-	●	●	●	●	●	●	●	●	●	●
(2, 5)	●	-	●	-	●	●	●	●	●	●	●	●	●
(3, 7, 8)	●	-	-	●	-	-	●	●	●	●	●	●	●

The Young diagram and Numerical Semigroup: Λ relative to the Numerical semigroup: from the top down, $1 \le i \le g$, the rows have length:

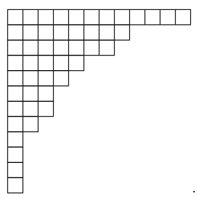
$$\Lambda_i = N(g) - N(i-1) - g + i - 1 = g - N(i-1) + (i-1),$$



Example: $(r, s) = (5, 7)$ (Table 3), we have

Table 3

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(i)$	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	x^4	y^3	x^3y	x^2y^2
$N(i)$	0	5	7	10	12	14	15	17	19	20	21	22	24
Λ_i	-	12	8	7	5	4	3	3	2	1	1	1	1



the genus of curves X**:** genus of (3,4,5) curve is 2.
Notation X**:** Let $\phi_n := \phi_n^{(2)}$, and $N_n := N_n^{(2)}$.

The canonical bundle: $\mathcal{K}_X$, $\{\nu^I{}_1, \nu^I{}_2\}$ of $H^0(X, \mathcal{K}_X)$,

$$\nu_1^I = \frac{\mathrm{d}x}{3y_5} \quad \text{and} \quad \nu_2^I = \frac{\mathrm{d}x}{3y_4}$$

The monomial $\phi_{H^1 i}$ **realated to** $H^1(X, \mathcal{O}_X(*\infty))$**:**

$\phi_{H^1 i} \in R$ such that $\dfrac{\phi_{H^1 i-1} \mathrm{d}x}{3y_4 y_5}$ is holomorphic in $X \setminus \infty$.

Table 4

	0	1	2	3	4	5	6	7	8	9
X_2	1	-	-	x	y_4	y_5	x^2	xy_4	xy_5	y_4y_5
ϕ_n	ϕ_0	-	-	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
$\phi_{H^1 n}$	-	-	-	$\phi_{H^1 0}$	$\phi_{H^1 1}$	-	$\phi_{H^1 2}$	$\phi_{H^1 3}$	$\phi_{H^1 4}$	$\phi_{H^1 5}$

The degree of $\phi_{H^1 i}$:
the degree is given by $N_{H^1,n} := \deg_{w^{-1}}(\phi_{H^1 n})$
The canonical divisor:
By letting $B_a := (b_a, 0, 0)$ $(a = 0, 1, 2)$,

$$(\nu_1^I) = \infty + B_0 \sim (\mathrm{d}x/y_5^2) = 2(3\infty - B_1 - B_2)$$
$$(\nu_1^I) \sim (\nu_2^I) = B_1 + B_2$$
$$\sim (\mathrm{d}x/y_4^2) = 2(2\infty - B_0) = 2(\infty + (\infty - B_0)).$$

The Homology bases $H_1(X, \mathbb{Z})$: α_i, β_j $(1 \leqq i, j \leqq 2)$ of such that their intersection numbers are $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{ij}$,.
The period matrixes $H_1(X, \mathbb{Z})$:

$$[\,\omega' \ \omega''] = \frac{1}{2}\left[\int_{\alpha_i} \nu_j^I \quad \int_{\beta_i} \nu_j^I\right]_{i,j=1,2}.$$

The non-shifted Abelian map: For $P \in X$,

$$\hat{u}_o : X \to \mathbb{C}^2, \quad \hat{u}_o(P) = \int_\infty^P \nu^I \in \mathbb{C}^2$$

The shifted Abelian map $\hat{u} : S^k X \to \mathbb{C}^2$: For $(P_1, \cdots, P_k) \in S^k X$,

$$\hat{u}(P_1, \cdots, P_k) := \hat{u}_o(P_1, \cdots, P_k) + \hat{u}_o(B_0)$$

The period lattice: $\Pi_2 := \langle \omega', \ \omega'' \rangle_{\mathbb{Z}}$.
The Jaobian $\mathcal{J}$:
κ $\mathbb{C}^g \to \mathcal{J} = \mathbb{C}^g/\Pi$,
Inverse image of the singular locus:

$$\mathcal{S}_m^n(X) := \{D \in \mathcal{S}^n(X) \mid \dim|D| \geq m\},$$

where $|D|$ is the complete linear system $w^{-1}(w(D))$.
$\mathcal{W}_m^n := w(\mathcal{S}_m^n(X))$.

1.1 Addition structures in R

Natural polynomial For $(P_i)_{i=1,\ldots,n} \in X \backslash \infty$,

$$\alpha_n(P) := \alpha_n(P; P_1, \ldots, P_n) = \sum_{i=0}^n a_i \phi_i(P) \in R$$

, where $a_i \in \mathbb{C}$ and $a_n = 1$ such that

$$\alpha_n(P_i) = 0.$$

1) $\mu_n(P) := \alpha_n(P)$ such that $\deg_w$ is the smallest in R.
2) $\mu_{H^1,n}(P) := \alpha_n(P)$ such that $\deg_w$ is the smallest in R and $\dfrac{\mu_{H^1,n}(P)dx}{3y_4y_5} \in H^1(X \setminus \infty, \mathcal{O}_X)$.

Lemma 1.1.

$$\mu_n(P) = \mu_n(P; P_1, \ldots, P_n)$$
$$= \lim_{P_i' \to P_i} \frac{\begin{vmatrix} 1 & \phi_1(P_1') & \phi_2(P_1') & \cdots & \phi_n(P_1') \\ 1 & \phi_1(P_2') & \phi_2(P_2') & \cdots & \phi_n(P_2') \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_n') & \phi_2(P_n') & \cdots & \phi_n(P_n') \\ 1 & \phi_1(P) & \phi_2(P) & \cdots & \phi_n(P) \end{vmatrix}}{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_{n-1}(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_{n-1}(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_n) & \phi_2(P_n) & \cdots & \phi_{n-1}(P_n) \end{vmatrix}}$$

$$\mu_{H^1,n}(P) = \mu_{H^1,n}(P; P_1, \ldots, P_n)$$
$$= \lim_{P_i' \to P_i} \frac{\begin{vmatrix} \phi_{H^1,0}(P_1') & \phi_{H^1,1}(P_1') & \cdots & \phi_{H^1,n}(P_1') \\ \phi_{H^1,0}(P_2') & \phi_{H^1,1}(P_2') & \cdots & \phi_{H^1,n}(P_2') \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{H^1,0}(P_n') & \phi_{H^1,1}(P_n') & \cdots & \phi_{H^1,n}(P_n') \\ \phi_{H^1,0}(P) & \phi_{H^1,1}(P) & \cdots & \phi_{H^1,n}(P) \end{vmatrix}}{\begin{vmatrix} \phi_{H^1,0}(P_1) & \phi_{H^1,1}(P_1) & \cdots & \phi_{H^1,n-1}(P_1) \\ \phi_{H^1,0}(P_2) & \phi_{H^1,1}(P_2) & \cdots & \phi_{H^1,n-1}(P_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{H^1,0}(P_n) & \phi_{H^1,1}(P_n) & \cdots & \phi_{H^1,n-1}(P_n) \end{vmatrix}}$$

generalized Mumford U

$$\mu_{H^1,1}(P) = \frac{\begin{vmatrix} y_{4,1} & y_{5,1} \\ y_4 & y_5 \end{vmatrix}}{y_{4,1}} = y_5 - \frac{y_{5,1}}{y_{4,1}} y_4$$

$$\mu_{H^1,2}(P) = \frac{\begin{vmatrix} y_{4,1} & y_{5,1} & xy_{4,1} \\ y_{4,2} & y_{5,2} & xy_{4,2} \\ y_4 & y_5 & xy_4 \end{vmatrix}}{\begin{vmatrix} y_{4,1} & y_{5,1} \\ y_{4,2} & y_{5,2} \\ y_4 & y_5 \end{vmatrix}}$$
$$= xy_4 - \frac{\begin{vmatrix} y_{4,1} & xy_{4,1} \\ y_{4,2} & xy_{4,2} \end{vmatrix}}{\begin{vmatrix} y_{4,1} & y_{5,1} \\ y_{4,2} & y_{5,2} \\ y_4 & y_5 \end{vmatrix}} y_5 + \frac{\begin{vmatrix} y_{5,1} & xy_{4,1} \\ y_{5,2} & xy_{4,2} \end{vmatrix}}{\begin{vmatrix} y_{4,1} & y_{5,1} \\ y_{4,2} & y_{5,2} \\ y_4 & y_5 \end{vmatrix}} y_4$$

$$\mu_n(P) = \phi_n(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \mu_{n,k}(P_1, \ldots, P_n) \phi_k(P),$$

with the convention $\mu_{n,n}(P_1, \ldots, P_n) \equiv 1$.

Lemma 1.2. *For $(P_i)_{i=1,\ldots,n} \in \mathcal{S}^n(X \backslash \infty)$, the function μ_n induces the map*

$$\mu_n^\# : \mathcal{S}^n(X \backslash \infty) \to \mathcal{S}^{N(n)-n}(X),$$

i.e., to $(P_i)_{i=1,\ldots,n} \in \mathcal{S}^n(X \backslash \infty)$ there corresponds an element $(Q_i)_{i=1,\ldots,N(n)-n} \in \mathcal{S}^{N(n)-n}(X)$, such that

$$\sum_{i=1}^n P_i - n\infty \sim -\sum_{i=1}^{N(n)-n} Q_i + (N(n) - n)\infty.$$

Lemma 1.3. *For $(P_i)_{i=1,\dots,n} \in \mathcal{S}^n(X\backslash\infty)$, the function $\mu_{H^1,n}$ induces the map*

$$\mu^{\#}_{H^1,n} : \mathcal{S}^1(X\backslash\infty) \to \mathcal{S}^{N_{H^1}(n)-2-n}(X),$$

i.e., to $(P_i)_{i=1,\dots,n} \in \mathcal{S}^n(X\backslash\infty)$ there corresponds an element $(Q_i)_{i=1,\dots,N_{H^1}(n)-2-n} \in \mathcal{S}^{N_{H^1}(n)-2-n}(X)$, such that

$$\sum_{i=1}^{n} P_i+B_0-(n+1)\infty \sim -(\sum_{i=1}^{N(n)-n} Q_i+B_0)+(N_{H^1}(n)-2-n)\infty.$$

For $P_1 \in X \setminus \infty$, there exists $Q_1 \in X$ such that

$$(P_1 + B_1) - 2\infty \sim -(Q_1 + B_1) + 2\infty.$$

Proposition 1.4.

$$\iota_n : \mathcal{W}^n \to \mathcal{W}^{N_{H^1}(n)-n}, \ \kappa \circ w \mapsto -\kappa \circ w.$$

Let image(ι_n) be denoted by $[-1]\mathcal{W}^n$.

The Serre involution on Pic^{g-1}, $\mathcal{L} \mapsto K_X\mathcal{L}^{-1}$, is given by ι_{g-1},

$$\iota_{g-1} : \mathcal{W}^{g-1} \to [-1]\mathcal{W}^{g-1}.$$

1.2 Differentials of the second and the third kinds

EEL-construction [5]:

This construction leads from a given commutative ring to the differentials of the second and third kind.

The fundamental normalized differential of the second kind in [7, Corollary 2.6]

Definition 1.5. *A two-form $\Omega(P_1, P_2)$ on $X \times X$ is called a* **fundamental differential of the second kind** *if it is symmetric, $\Omega(P_1, P_2) = \Omega(P_2, P_1)$, it has its only pole (of second order) along the diagonal of $X \times X$, and in the vicinity of each point (P_1, P_2) is expanded in power series as*

$$\Omega(P_1, P_2) = \Big(\frac{1}{(t_{P_1} - t'_{P_2})^2} + \mathrm{d}_{\geq}(1)\Big)\mathrm{d}t_{P_1} \otimes \mathrm{d}t_{P_2} \quad (\text{as } P_1 \to P_2)$$

where t_P is a local coordinate at a point $P \in X$.

Conventions on Notations For $P_a \in X$, P_a is represented by $(x_a, y_{4,a}, y_{5,a})$ or $(x_{P_a}, y_{4,P_a}, y_{5,P_a})$ and for $P \in X$, P is expressed by (x, y_4, y_5).

Differentials of the second kind

Proposition 1.6. *By letting*

$$\Sigma(P,Q) := \frac{y_{4,P}y_{5,P} + y_{4,P}y_{5,Q} + y_{4,Q}y_{5,P}}{(x_P - x_Q)3y_{4,P}y_{5,P}}\mathrm{d}x_P$$

$\Sigma(P,Q)$ has the properties:
1) $\Sigma(P,Q)$ as a function of P is singular at $Q = (x_Q, y_{4,Q}, y_{5,Q})$ and ∞, and vanishes at $\hat{\zeta}_3^\ell(Q) = (x_Q, \zeta_3^\ell y_{4,Q}, \zeta_3^{2\ell} y_{5,Q})$, $(\ell = 1, 2)$, and
2) $\Sigma(P,Q)$ as a function of Q is singular at P and at ∞.

Proof. Direct computations lead the results. □

Proposition 1.7. *There exist differentials $\nu_j^{II} = \nu_j^{II}(x, y_4, y_5)$ $(j = 1, 2)$ of the second kind such that they have their only pole at ∞ and satisfy the relation,*

$$\mathrm{d}_Q\Sigma(P,Q) - \mathrm{d}_P\Sigma(Q,P) = \sum_{i=1}^{2}\Big(\nu_i^I(Q)\otimes\nu_i^{II}(P) - \nu_i^I(P)\otimes\nu_i^{II}(Q)\Big)$$

where $\mathrm{d}_Q\Sigma(P,Q) := \mathrm{d}x_P \otimes \mathrm{d}x_Q \dfrac{\partial}{\partial x_Q}\dfrac{\Sigma(P,Q)}{\mathrm{d}x_P}.$

The differentials $\{\nu_1^{II}, \nu_2^{II}\}$ are determined modulo the $\mathbb{C}$-linear space spanned by $\langle\nu_j^I\rangle_{j=1,2}$; we fix

$$\{\nu_1^{II}, \nu_2^{II}\} = \left\{\frac{-\left(2x + \lambda_1^{(2)}\right)\mathrm{d}x}{3y_4}, \ \frac{-x\mathrm{d}x}{3y_5}\right\} \in H^1(X \setminus \infty, \mathcal{O}_X)$$

as their representative.

Proof. Hard direct computations lead the results. □

Corollary 1.8. *1) The one form, $\Pi_{P_1}^{P_2}(P) := \Sigma(P, P_1) - \Sigma(P, P_2)$, is a differential of the third kind, whose only (first-order) poles are $P = P_1$ and $P = P_2$, and residues $+1$ and -1 respectively.*

2) $\Omega(P_1, P_2) := \mathrm{d}_{P_2}\Sigma(P_1, P_2) + \sum_{i=1}^{2}\nu_i^I(P_1) \otimes \nu_i^{II}(P_2)$

$$\Omega(P_1, P_2) = \frac{F(P_1, P_2)\mathrm{d}x_1 \otimes \mathrm{d}x_2}{(x_{P_1} - x_{P_2})^2 9y_{4,P_1}y_{5,P_1}y_{4,P_2}y_{5,P_2}}$$

where F is an element of $R \otimes R$.

Proof. Direct computations give the claims. □

Lemma 1.9.

$$\lim_{P_1\to\infty}\frac{F(P_1, P_2)}{\phi_{H^1 1}(P_1)(x_{P_1} - x_{P_2})^2} = \phi_{H^1 2}(P_2) = x_{P_2}y_{4,P_2}.$$

Proof. B_2 in the proof of Proposition 1.7 leads the result. □

Integrals:

$$\Omega_{Q_1,Q_2}^{P_1,P_2} := \int_{P_2}^{P_1}\int_{Q_2}^{Q_1}\Omega(P,Q)$$

$$= \int_{P_2}^{P_1}(\Sigma(P,Q_1) - \Sigma(P,Q_2)) + \sum_{i=1}^{4}\int_{P_2}^{P_1}\nu_i^I(P)\int_{Q_2}^{Q_1}\nu_i^{II}(P).$$

2 The sigma function for (3, 4, 5) curve

2.1 Generalized Legendre relation

Corresponding to the complete integral of the first kind, we define the complete integral of the second kind,

$$[\eta' \ \eta''] := \frac{1}{2}\left[\int_{\alpha_i}\nu_j^{II} \quad \int_{\beta_i}\nu_j^{II}\right]_{i,j=1,2}.$$

Let τ_{Q_1,Q_2} be the normalized differential of the third kind such that τ_{Q_1,Q_2} has residues $+1$ and -1 at Q_1 and Q_2 respectively, is regular everywhere else, and is normalized, $\int_{\alpha_i}\tau_{P,Q} = 0$ for $i = 1, 2$ [7, p.4]. The following Lemma corresponding to Corollary 2.6 (ii) in [7] holds:

Lemma 2.1. *By letting* $\gamma = {\omega'}^{-1}\eta'$, *we have*

$$\Omega^{P_1,P_2}_{Q_1,Q_2} = \int_{P_2}^{P_1} \tau_{Q_1,Q_2} + \sum_{i,j=1}^{2} \gamma_{ij} \int_{P_2}^{P_1} \nu^I_i \int_{Q_2}^{Q_1} \nu^I_j.$$

Proof. The same as [20, I: Lemma 4.1]. ☐

The following Proposition provides a symplectic structure in the Jacobian $\mathcal{J}_2$, known as *generalized Legendre relation* [3, 4, 20]:

Proposition 2.2. $M \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} {}^tM = 2\pi\sqrt{-1} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ *for* $M := \begin{bmatrix} 2\omega' & 2\omega'' \\ 2\eta' & 2\eta'' \end{bmatrix}$.

Proof. The same as [20, I: Propositon 4.2]. ☐

2.2 The σ function

Due to the Riemann relations [7], $\mathrm{Im}\,({\omega'}^{-1}\omega'')$ is positive definite. Theorem 1.1 in [7] gives $\delta := \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} \in \left(\dfrac{\mathbb{Z}}{2}\right)^4$ be the theta characteristic which is equal to the Riemann constant ξ_R and the period matrix $[\,2\omega'\ 2\omega'']$. We note that $\xi_R = \hat{u}(P_R)$ for a point $P_R \in X$ satisfying $2P_R + 2B_0 - 4\infty \sim 0$. We define an entire function of (a column-vector) $u = {}^t(u_1, u_2) \in \mathbb{C}^2$,

$$\sigma(u) = c\mathrm{e}^{-\frac{1}{2}\ {}^tu\eta'{\omega'}^{-1}\ u} \times \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{\left[\pi\sqrt{-1}\left\{\ {}^t(n+\delta'')\omega'^{-1}\omega''(n+\delta'')+\ {}^t(n+\delta'')(\omega'^{-1}u+\delta')\right\}\right]}$$

where c is a certain constant as in (2.1).

For a given $u \in \mathbb{C}^2$, we introduce u' and u'' in $\mathbb{R}^2$ so that $u = 2\omega' u' + 2\omega'' u''$.

Proposition 2.3. *For* u, $v \in \mathbb{C}^2$, *and* ℓ $(= 2\omega'\ell' + 2\omega''\ell'')$ $\in \Pi_2$, *by letting* $L(u, v) := 2\ {}^tu(\eta' v' + \eta'' v'')$, $\chi(\ell) := \exp[\pi\sqrt{-1}\big(2({}^t\ell''\delta' - {}^t\ell'\delta'') + {}^t\ell'\ell''\big)]$, *we have a translational relation,*

$$\sigma(u + \ell) = \sigma(u)\exp(L(u + \frac{1}{2}\ell, \ell))\chi(\ell).$$

Proof. The same as [20, I: Prop.4.3]. ☐

The vanishing locus of σ is simply given by $\Theta^1 := (\mathcal{W}^1 \cup [-1]\mathcal{W}^1) = \mathcal{W}^1$.

2.3 The Riemann fundamental relation

As in [20, I: Prop4.4], we have the Riemann fundamental relation:

Proposition 2.4. *For* $(P, Q, P_i, P'_i) \in X^2 \times (S^2(X) \setminus S^2_1(X)) \times (S^2(X) \setminus S^2_1(X))$,

$$\exp\left(\sum_{i,j=1}^{2} \Omega^{P,Q}_{P_i,P'_j}\right) = \frac{\sigma(w_o(P) - w(P_1, P_2))\sigma(w_o(Q) - w(P'_1, P'_2))}{\sigma((w_o(Q) - w(P_1, P_2))\sigma(w_o(P) - w(P'_1, P'_2))}.$$

Using the differential identity,

$$\sum_{i,j=1}^{2} \phi_{H^1{}_{i-1}}(P'_1)\phi_{H^1{}_{j-1}}(P'_2)\frac{\partial^2}{\partial\hat{u}_i(P'_1)\partial\hat{u}_j(P'_2)} = 9y_{4,P'_1}y_{5,P'_1}y_{4,P'_2}y_{5,P'_2}\frac{\partial^2}{\partial x'_1\partial x'_2}$$

Proposition 2.5. *For* $(P, P_1, P_2) \in X \times S^2(X) \setminus S^2_1(X)$ *and* $u := w(P_1, P_2)$, *the equality*

$$\sum_{i,j=1}^{2} \wp_{i,j}\left(w_o(P) - u\right)\phi_{H^1{}_{i-1}}(P)\phi_{H^1{}_{j-1}}(P_a) = \frac{F(P, P_a)}{(x - x_a)^2}$$

holds for every $a = 1, 2$, *where we set*

$$\wp_{ij}(u) := -\frac{\sigma_i(u)\sigma_j(u) - \sigma(u)\sigma_{ij}(u)}{\sigma(u)^2} \equiv -\frac{\partial^2}{\partial u_i\partial u_j}\log\sigma(u).$$

2.4 Jacobi inversion formulae

Theorem 2.6. *1) For* $(P, P_1, P_2) \in X \times \big(S^2(X) \setminus S^2_1(X)\big)$, *we have*

1-1) $\mu_2(P; P_1, P_2) = xy_4 - \wp_{22}(w(P_1, P_2))y_4 + \wp_{21}(w(P_1, P_2))y_5$.

1-2) $$\wp_{22}(w(P_1, P_2)) = \frac{y_{4,1}x_2y_{4,2} - y_{4,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}}$$
$$\wp_{21}(w(P_1, P_2)) = \frac{y_{5,1}x_2y_{4,2} - y_{5,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}}.$$

2) For $(P, P_1) \in X \times (X \setminus S^1_1(X))$ *and* $u = w(P_1) \in \kappa^{-1}(\mathcal{W}^1)$,

$$\mu_1(P; P_1) = y_5 - \frac{\sigma_1(u)}{\sigma_2(u)}y_4, \quad \textit{and} \quad \frac{\sigma_1(u)}{\sigma_2(u)} = \frac{y_5}{y_4}.$$

Proof. 1) is the same as [20, I: Prop. 4.6]. As in [20, I: Theorem 5.1], by considering $\lim_{P_2\to\infty}\dfrac{\wp_{21}(\hat{u}(P_1, P_2))}{\wp_{22}(\hat{u}(P_1, P_2))}$, we have the second result. ☐

Following the statement by Buchstaber, Leykin and Enolskii, Nakayashiki showed that the leading of the sigma function for (r, s) curve is expressed by Schur function [21]. Noting (**??**) and degrees of u, the above Jacobi inversion formulae gives an extension that

$$\sigma(u) = \frac{1}{2}u_2^2 - u_1 + \sum_{|\alpha|>2} a_\alpha u^\alpha$$

where $a_\alpha \in \mathbb{Q}[b_1, \cdots, b_5]$, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$ and $u^\alpha = u_1^{\alpha_1}u_2^{\alpha_2}$. The prefactor c is determined by this relation. Since for a Young diagram Λ, S_Λ and s_Λ are the Schur functions defined by

$$S_\Lambda(T_1, T_2) = t_1t_2 = \frac{1}{2}T_1^2 - T_2 \qquad (2.1)$$

where $T_1 := t_1 + t_2$ and $T_2 := \frac{1}{2}(t_1^2 + t_2^2)$, we have

$$\sigma(u) = S_\Lambda(u_1, u_2) + \sum_{|\alpha|>2} a_\alpha u^\alpha.$$

3 Observation to Norton Number

3.1 Norton condition

Let $\mathcal{A}$ be a $\mathbb{Q}$ ring, with filtration assosicated to multiplication, $\mathcal{A} = \cup_i \mathcal{A}_i$ and $\mathcal{A}_j\mathcal{A}_i \subset \mathcal{A}_{i+j}$. Let us consider $q = \mathrm{e}^{2\pi\sqrt{-1}\tau}$ for $z \in H_+ := \{\tau \in \mathbb{C} \mid \Im\tau \geq 0\}$ and a function

$$f(q) = q^{-1} + h_1 q + h_2 q^2 + \cdots$$

where $h_j \in \mathcal{A}_j$. We also write $f(\tau) = f(q)$.

The Grunsky coefficient $h_{m,n} \in \mathcal{A}_{m+n-1}$ of $f(q)$ is defined by [?]

$$\sum_{m,n} h_{m,n} p^m q^n := \log\left(pq\frac{f(q) - f(p)}{p - q}\right)$$

and the Faber polynomial $F_{f,n}(f)$ [?, 7] is defined by

$$F_{f,n}(f(q)) = \frac{1}{p^n} + n\sum_{m>0} h_{m,n} p^m,$$

where $h_{1,m} = h_m$. From the definition of Grunsky coefficients, we have these property:

Lemma 3.1.

$$h_{r,s} = h_{r+s-1} + \frac{1}{r+s}\sum_{m=1}^{r-1}\sum_{m=1}^{s-1}(n+m)h_{r+s-m-n-1}h_{m,n}.$$

These appear in the dispersionless KP hierarchy [?].

The replicable functions are generalizations of the the elliptic modular function $j(\tau)$, which is characterized by its expansion and the following property under the action of Hecke operators T_n for every $n \geq 1$,

$$nT_n(j(\tau)) = \sum_{ad=n, 0\leq b<d} j\left(\frac{a\tau + b}{d}\right) = F_{f,n}(j(\tau)).$$

This gives an $SL_2(\mathbb{Z})$ action on j.

The replicable functions were characterized by Norton, as having certain properties under the action of the Hecke operators, as members of the finite family of functions $\{f^{(a)}\}$ given by

$$\sum_{ad=n, 0\leq b<d} f^{(a)}\left(\frac{a\tau + b}{d}\right) = F_{f,n}(f(\tau)).$$

Norton considered characterized the coefficients $h_{m,1}$ due to such action.

Definition 3.2. *If whenever $nm = rs$ and $(n, m) = (r, s)$, we have the identity $h_{n,m} = h_{r,s}$, we say that $\mathcal{H}$ is replicable. The condition is called* Norton condition.

Example 3.3.
1. $h_6 = h_{3,2} = h_4 + h_1 h_2$,
2. $h_{12} = h_{3,4} = h_6 + h_1^2 h_2 + 2h_2 h_3 + h_1 h_4$,
3. $h_{10} = h_{5,2} = h_6 + h_1 h_4 + h_2 h_3$,
4. $h_{14} = h_{7,2} = h_8 + h_1 h_6 + h_2 h_5 + h_3 h_4$, and
5. $h_{15} = h_{3,5} = h_7 + 2h_2 h_4 + h_3^2 + h_5 h_1 + h_1^2 h_3 + h_1 h_2^2$.

The Norton condition is not consistent with the grading of $\mathcal{A}$ unless we modify the grading, e.g., $h_{-1} = 1$ with the weight -1.

Norton proved that

Theorem 3.4. *For a replicable function f with Fourier coefficients h_ℓ, every h_n belongs to $\mathbb{Q}[\mathcal{H}]$, where $\mathcal{H} := \{h_i \mid i \in \Phi\}$ and $\Phi := \{1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23\}$.*

Table 4

	0	1	2	3	4	5	6	7	8	9	10	11	12
(3, 7, 8)	●	-	-	●	-	-	●	●	●	●	●	●	●
(6, 13, 14, 15, 16)	●	-	-	-	-	-	●	-	-	-	-	-	●
Norton No.	-	○	○	○	○	○	-	○	○	○	-	○	-
	13	14	15	16	17	18	19	20	21	22	23	24	25
(3, 7, 8)	●	●	●	●	●	●	●	●	●	●	●	●	●
(6, 13, 14, 15, 16)	●	●	●	●	-	●	●	●	●	●	-	●	●
Norton No.	-	-	-	-	○	-	○	-	-	-	○	-	-

$$L(H_{12}) := \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\},$$

while the Norton sequence is given by [?, ?, 22, ?] (see Appendix A),

$$\{1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23\}.$$

Komeda's construction:

Proposition 3.5. *For the numerical semigroup $\langle 6, 13, 14, 15, 16\rangle$, there is an algebraic curve which has a point with Weierstrass non-gap given by the semigroup.*

We construct the curve X_{12} and its sigma function:

Remark 3.6.
1. As mentioned in Remark **??**, the $\sigma^{(3)}$ function over $\mathbb{C}^{12}$ is an entire function and is also expected to be expressed by the Schur function. If the moduli parameters $(b_1, \ldots, b_7)$ are polynomials of p and q, it is expected that
$$\sigma^{(3)}(u) - S_{\mathcal{Y}_{12}}(T)|_{T_{(i)} = t_i} = \sum_{m,n} \hat{h}_{m,n} q^m p^n,$$
where
$$\hat{h}_{m,n} \in \mathbb{Q}[\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3, \mathsf{t}_4, \mathsf{t}_5, \mathsf{t}_7, \mathsf{t}_8, \mathsf{t}_9, \mathsf{t}_{10}, \mathsf{t}_{11}, \mathsf{t}_{17}, \mathsf{t}_{23}].$$
2. $\sigma^{(3)}$ is a generalization of Weierstrass sigma function whereas the Weierstrass sigma function plays an important role in [8]; if we consider a suitable degeneration of the curve X_{12} associated with elliptic curves, it is expected that $\sigma^{(3)}$ might be written in terms of Weierstrass' elliptic σ.
3. By the Riemann-Kempf theorem, using (**??**), a suitable derivative of σ, $\sigma_{\mathsf{t}_{i_1}, \cdots, \mathsf{t}_{i_k}}$, is a function of $\{\mathsf{t}_1, \mathsf{t}_2, \cdots, \mathsf{t}_{23-N^{(12)}(13-k)}\}$ over an open subset of the subvariety $\kappa^{-1}\mathcal{W}_k$ [?]. The other t's of the hierarchy are functions of $\mathsf{t}_1, \mathsf{t}_2, \cdots, \mathsf{t}_{23-N^{(12)}(13-k)}$ [?]. This is analogous to the fact that for a suitable replicable function, every c_i belongs to a subring $\mathbb{Q}[h_1, \ldots, h_\ell]$ of $\mathbb{Q}[h_1, h_2, h_3, h_4, h_5, h_7, h_8, h_9, h_{11}, h_{17}, h_{19}, h_{23}]$, and a subset of h's are functions of the other h's, e.g., for the case of the j-function, $c_i \in \mathbb{Q}[h_1, h_2, h_3, h_5]$ [?].
4. For an (n, s) curve, algebraic solutions to the dKP equation exist [?]; it should be possible to generalize them to curves covered locally by affine patches such as X_4 and X_{12}. As an illustration, let us consider the case that

$\hat{k}_2 = k_2$: Then $\phi_{11}^{(12)}\phi_9^{(12)} = (\phi_{10}^{(12)})^2$ and for $v := y_{14}/y_{15} = \frac{d}{d\mathsf{t}_2}\frac{\sigma_1^{(3)}}{\sigma_2^{(3)}} \Big/ \frac{d}{d\mathsf{t}_3}\frac{\sigma_1^{(3)}}{\sigma_2^{(3)}}$ over $\hat{u}(X_{12}) \subset \mathbb{C}^{12}$, we have a X_{12} curve solution of the dKP equation,

$$\frac{\partial}{\partial \mathsf{t}_3}\frac{\partial}{\partial \mathsf{t}_1}v + \frac{\partial}{\partial \mathsf{t}_1}\left(v\frac{\partial}{\partial \mathsf{t}_1}v\right) - 2\frac{\partial}{\partial \mathsf{t}_2}\frac{\partial}{\partial \mathsf{t}_2}v = 0.$$

The proof is the same as in [**?**]. It is expected that we might have the dKP hierarchy for $(\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3, \mathsf{t}_4, \mathsf{t}_5, \mathsf{t}_7, \mathsf{t}_8, \mathsf{t}_9, \mathsf{t}_{10}, \mathsf{t}_{11}, \mathsf{t}_{17}, \mathsf{t}_{23})$. It is noted that J. McKay, and A. Sebbar considered the relation between replicable functions and τ-functions of the dKP hierarchy [22].

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