Truncated Young diagrams and sigma functions

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March 22, 2012

Abstract

For a (r,s) plane curve, a symmetric Young diagram is induced. The sigma function of the curve has the Schur function as the leading term in its expansion around the origin. When we decompose the Young diagram into two parts, upper side truncated one and lower side one, there appear natural sigma functions related to these truncated Young diagrams. For the lower side truncated one, it corresponds to the sigma function of strata of the Jacobian whereas the other one does to sigma function for a space curve.

1 Notations

Cyclic (r, s) curve

$$X := \{(x, y) \mid y^r = f(x)\} \cup \infty$$

$$f(x, y) := y^r - (x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_1 x + \lambda_0). \qquad (1.1)$$

The integers r and s such that that $(r, s) = 1$ and $r < s$; the complex numbers $\lambda_0, \ldots, \lambda_{s-1}$ such that the finite part of X is smooth.
The genus: $g = \frac{(r-1)(s-1)}{2}$.
The commutative ring $R = \mathcal{O}_X(*\infty)$. $R := \mathbb{C}[x, y]/(f(x, y))$:
the sheaf of holomorphic functions over X $\mathcal{O}_X$:
the Jacobian of X**:** $\mathcal{J}$:
the (monic) monomial for a non-negative integer n $\phi_n \in R$:

$$R = \oplus_{n=0}\mathbb{C}\phi_n, \text{ as a vector space}$$

the order of pole at ∞ $N(n) := \deg_w(\phi_n)$ such that $N(n) < N(n + 1)$.

$$\phi_0 = 1, \phi_1 = x, etc.;$$

$$N(0) = 0, N(g - 1) = 2g - 2, N(g) = 2g.$$

w-degree: $\deg_w : R \to \mathbb{Z}$, $\deg_w(x) = r$, $\deg_w(y) = s$, $\deg_w(\phi_n(P)) = N(n)$.

$R_\lambda := \mathbb{Q}[x, y, \lambda_0, \ldots, \lambda_{s-1}]/(f(x, y))$ λ-degree, $\mathrm{wdeg}_\lambda : R_\lambda \to \mathbb{Z}$ as an extension of the w-degree by assigning the degree $(s - i)r$ to each λ_i.
Affine coordinate: $P \in X\backslash\infty$ by its affine coordinates (x, y);

$(P_1, \ldots, P_k)$, or by a divisor $D = \sum_{i=1}^k P_i$, an element of $\mathcal{S}^k(X)$, the k-th symmetric product of the curve.
The local parameter t_∞ at ∞,

$$x = \frac{1}{t_\infty^r}, \; y = \frac{1}{t_\infty^s}\left(1 + d_>(t_\infty)\right),$$

$$\phi_n(P) = \frac{1}{t_\infty^{N(n)}}\left(1 + d_>(t_\infty)\right).$$

The canonical bundle: K_X, $\{{\nu^I}_1, \ldots, {\nu^I}_g\}$ of $H^0(X, K_X)$,

$${\nu^I}_i = \frac{\phi_{i-1}(P)dx}{ry^{r-1}}, \; (i = 1, \ldots, g),$$

$${\nu^I}_i = t_\infty^{2g-N(i-1)-2}(1 + d_{>0}(t_\infty))dt_\infty,$$

the degree is given by $\deg_{w^{-1}}({\nu^I}_i) = 2g - N(i-1) - 2$, where $\deg_{w^{-1}}(f) = -\deg_w(f)$,
The Abel images:

$$\mathcal{W}^k := \kappa\left(\left\{\sum_{i=1}^k \int_\infty^{(x_i,y_i)} \begin{pmatrix} {\nu^I}_1 \\ \vdots \\ {\nu^I}_g \end{pmatrix} \Big| (x_i, y_i) \in X\right\}\right) \subset \mathcal{J},$$

where κ is the projection $\mathbb{C}^g \to \mathcal{J} = \mathbb{C}^g/\Pi$,
The period lattice: Π of the basis $\{{\nu^I}_1, \ldots, {\nu^I}_g\}$,
Abel Map: $w : (P_1, \ldots, P_k) \mapsto w(P_1, \ldots, P_k) = \sum_{i=1}^k \int_\infty^{P_i} \nu^I \in \mathbb{C}^g$,

$$\mathcal{S}^n_m(X) := \{D \in \mathcal{S}^n(X) \mid \dim|D| \geq m\},$$

where $|D|$ is the complete linear system $w^{-1}(w(D))$. $\mathcal{W}^n_m := w(\mathcal{S}^n_m(X))$.

The Young diagram: Λ relative to the Weierstrass-gap sequence: from the top down, $1 \leq i \leq g$, the rows have length:

$$\Lambda_i = N(g) - N(i - 1) - g + i - 1 = g - N(i - 1) + (i - 1),$$

$$|\Lambda| = \sum_{i=1}^g \Lambda_i = \frac{1}{24}(r^2 - 1)(s^2 - 1) = g + \mathrm{w}(\infty),$$

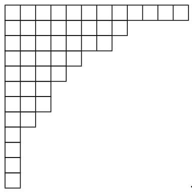
The characteristics of the partition Let a_i and b_i be the number of boxes below and to the right of the i-th box of the diagonal, reading from lower right $\Lambda = (a_1, \ldots, a_r; b_1, \ldots, b_r)$, $a_i < a_j$ and $b_i < b_j$ for $i < j$. $|\Lambda| = \sum_{i=1}^r (a_i + b_i + 1)$.

$$(7, 5, 4, 2, 1) = (0, 2, 4; 1, 3, 6) = \begin{array}{|c|c|c|c|c|c|c|} \hline - & 1 & 2 & 3 & 4 & 5 & b_3 \\ \hline 1 & - & 1 & 2 & b_2 \\ \cline{1-5} 2 & 1 & - & b_1 \\ \cline{1-4} 3 & a_2 \\ \cline{1-2} 4 \\ \cline{1-1} a_3 \\ \cline{1-1} \end{array} .$$

Example: $(r, s) = (5, 7)$ (Table 2.1), we have

Table 2.1

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(i)$	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	x^4	y^3	x^3y	x^2y^2
$N(i)$	0	5	7	10	12	14	15	17	19	20	21	22	24
Λ_i	-	12	8	7	5	4	3	3	2	1	1	1	1



Lemma 1.1. *For $v \in w(P)$, $P \in X$,*

$$\deg_{w^{-1}}(v_i) = N(g) - N(i-1) - 1$$
$$= 2g - N(i-1) - 1 = \Lambda_i + g - i,$$

and $\deg_{w^{-1}}(v_g) = 1$.

The multi-index convention for $\alpha := (\alpha_1, \dots, \alpha_k)$,

$$t^\alpha := t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k}, \ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_k),$$
$$|\alpha| := \sum_{i=1}^{k} \alpha_i,$$

and extend the definition $d_>(t_i^\ell)$ to the variables, $t_1, \dots, t_k$: $d_>(t^\ell) \in \{\sum_{|\alpha|>\ell} a_\alpha t^\alpha\}$ and $d_\geq(t^\ell) \in \{\sum_{|\alpha|\geq \ell} a_\alpha t^\alpha\}$.

ℓ-reduced Frobenius-Stickelberger (FS) matrix: $P_1, \dots, P_n$ be in $X \backslash \infty$.

$$\Psi_n^{(\check\ell)}(P_1, P_2, \dots, P_n) :=$$
$$\begin{pmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \check\phi_\ell(P_1) & \cdots & \phi_n(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \check\phi_\ell(P_2) & \cdots & \phi_n(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_n) & \phi_2(P_n) & \cdots & \check\phi_\ell(P_n) & \cdots & \phi_n(P_n) \end{pmatrix},$$

$$\psi_n^{(\check\ell)}(P_1, P_2, \dots, P_n) := \det(\Psi_n^{(\check\ell)}(P_1, P_2, \dots, P_n))$$

We call this matrix *Frobenius-Stickelberger (FS) matrix* and its determinant *Frobenius-Stickelberger (FS) determinant.*

$$\psi_n(P_1, \dots, P_n) := |\Psi_n^{(\check n)}(P_1, \dots, P_n)|,$$
$$\Psi_n(P_1, \dots, P_n) := \Psi_n^{(\check n)}(P_1, \dots, P_n).$$

Addition structure: For n points $(P_i)_{i=1,\dots,n} \in X\backslash\infty$, we find an element of R associated with any point $P = (x, y)$ in $(X\backslash\infty)$, $\mu_n(P) := \mu_n(P; P_1, \dots, P_n) = \sum_{i=0}^n a_i \phi_i(P)$, $a_i \in \mathbb{C}$ and $a_n = 1$, which has a zero at each point P_i (with multiplicity, if the P_i are repeated) and has smallest possible order of pole at ∞ with this property.

Lemma 1.2.

$$\mu_n(P) = \mu_n(P; P_1, \dots, P_n)$$
$$:= \lim_{P_i' \to P_i} \frac{1}{\psi_n(P_1', \dots, P_n')} \psi_{n+1}(P_1', \dots, P_n', P),$$

generalized Mumford U $\mu_{n,k}(P_1, \dots, P_n)$

$$\mu_n(P) = \phi_n(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \mu_{n,k}(P_1, \dots, P_n) \phi_k(P),$$

with the convention $\mu_{n,n}(P_1, \dots, P_n) \equiv 1$.

Lemma 1.3. *Let n be a positive integer. For $(P_i)_{i=1,\dots,n} \in \mathcal{S}^n(X\backslash\infty)$, the function μ_n over X induces the map (which we call by the same name):*

$$\mu_n : \mathcal{S}^n(X\backslash\infty) \to \mathcal{S}^{N(n)-n}(X),$$

i.e., to $(P_i)_{i=1,\dots,n} \in \mathcal{S}^n(X\backslash\infty)$ there corresponds an element $(Q_i)_{i=1,\dots,N(n)-n} \in \mathcal{S}^{N(n)-n}(X)$, such that

$$\sum_{i=1}^{n} P_i - n\infty \sim - \sum_{i=1}^{N(n)-n} Q_i + (N(n) - n)\infty.$$

Proposition 1.4. *For a positive integer, the Abel map composed with μ_n induces*

$$\iota_n : \mathcal{W}^n \to \mathcal{W}^{N(n)-n}, \ \kappa \circ w \mapsto -\kappa \circ w.$$

Let image(ι_n) be denoted by $[-1]\mathcal{W}^n$.

The Serre involution on Pic^{g-1}, $\mathcal{L} \mapsto K_X \mathcal{L}^{-1}$, is given by ι_{g-1},

$$\iota_{g-1} : \mathcal{W}^{g-1} \to [-1]\mathcal{W}^{g-1}.$$

2 The σ-function

$$\Theta^k := \mathcal{W}^k \cup [-1]\mathcal{W}^k, \ \ \Theta_1^k := w(\mathcal{S}_1^k(X)) \cup [-1]w(\mathcal{S}_1^k(X)).$$

The Schur function $\mathbf{s}_\Lambda(t)$:

$$\mathbf{s}_\Lambda(t) := \frac{|t_j^{\Lambda_i + g - i}|_{1 \leq i,j \leq g}}{|t_j^{i-1}|_{1 \leq i,j \leq g}}.$$

The complete homogeneous symmetric function $h_n^{\langle \ell_1, \ell_2 \rangle} = h_n(t_{\ell_1}, \dots, t_{\ell_2})$ for positive integers ℓ_1 and ℓ_2 $(\ell_1 < \ell_2)$:

$$\prod_{i=\ell_1}^{\ell_2} \frac{1}{(1 - z t_i)} = \sum_{n \geq 0} h_n^{\langle \ell_1, \ell_2 \rangle} z^n, \quad h_n^{\langle \ell_1, \ell_2 \rangle} = 0 \text{ for } n < 0.$$

$h_0 = 1$, $h_{i<0} = 0$ and $T_k := T_k^{\langle 1, g \rangle}$,

Power Symmetric Function: $T_k^{\langle \ell_1, \ell_2 \rangle} := \frac{1}{k} \sum_{j=\ell_1}^{\ell_2} t_j^k$.

Jacobi-Trudi Determinant: $\mathbf{s}_\Lambda(t) := |h_{\Lambda_i + j - i}|$

$$h_n = \frac{1}{n!} \begin{vmatrix} T_1 & -1 & 0 & \cdots & \\ 2T_2 & T_1 & -2 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n-1)T_{n-1} & (n-2)T_{n-2} & (n-3)T_{n-3} & \cdots & 1-n \\ nT_n & (n-1)T_{n-1} & (n-2)T_{n-2} & \cdots & T_1 \end{vmatrix}.$$

modified Jacobi-Trudi Determinant:

$\mathbf{s}_\Lambda(t) := |h_{\Lambda_i + j - i}^{\langle j, g \rangle}|_{1 \leq i,j \leq g}$

sigma and prime form (Nakayashiki)

$$\sigma(w(P_1, \dots, P_n)) = \frac{\prod_{i=1}^n \mathcal{E}(\infty, P_i)^n}{\prod_{i<j} \mathcal{E}(P_i, P_j)} \psi(P_1, \dots, P_n)$$

where $\mathcal{E}(P, Q)$ is the modified prime form.

Proposition 2.1. (Nakayashiki, Buchtaber-Enolskii-Leykin) *The expansion of $\sigma(u)$ at the origin takes the form*

$$\sigma(u) = S_\Lambda(T)|_{T_{\Lambda_i + g - i} = u_i} + \sum_{|\mathrm{w}_g(\alpha)| > |\Lambda|} c_\alpha u^\alpha$$

where $c_\alpha \in \mathbb{Q}[\lambda_j]$ and $S_\Lambda(T)$ is the lowest-order term in the w-degree of the u_i; $\sigma(u)$ is homogeneous of degree $|\Lambda|$ with respect to the λ-degrees.

We note that S_Λ is a function of $\{T_{\Lambda_i+g-i}\}_{i=1,\dots,g}$, even though *a priori* it depends on $\{T_i\}_{i=1,\dots,2g-1}$.

3 Algebraic expression of the Jacobian of a coordinate change

Let k be a positive integer $\le g$. K_k of $\{1,2,\dots,g\}$ with k elements.
relabeling $\iota:\{1,2,\dots,k\}\to K_k$ such that $\iota(i)<\iota(i+1)$ for $i=1,\dots,k-1$; $\iota'(i):=\iota(i)-1$.

$$\mathrm{proj}_{K_k}\circ w:(P_1,\dots,P_k)\mapsto (u_j=\sum_{i=1}^{k}\int_\infty^{P_i}\nu^{\mathrm{I}}_j)_{j\in K_k},$$

$$\begin{pmatrix}\partial_{u_{\iota(1)}}\\ \partial_{u_{\iota(2)}}\\ \vdots\\ \partial_{u_{\iota(k)}}\end{pmatrix}=r\begin{pmatrix}\phi_{\iota'(1)}(P_1)\cdots\phi_{\iota'(k)}(P_1)\\ \phi_{\iota'(1)}(P_2)\cdots\phi_{\iota'(k)}(P_2)\\ \vdots\ \ \ddots\ \ \vdots\\ \phi_{\iota'(1)}(P_k)\cdots\phi_{\iota'(k)}(P_k)\end{pmatrix}^{-1}\begin{pmatrix}y_1^{r-1}\partial_{x_1}\\ y_2^{r-1}\partial_{x_2}\\ \vdots\\ y_k^{r-1}\partial_{x_k}\end{pmatrix}.$$

4 Vanishing of σ on Θ^k $(0<k<g)$

The truncated Young diagrams:

$$\Lambda^{(k)}:=(\Lambda_1,\dots,\Lambda_k),\quad \Lambda^{[k]}:=(\Lambda_{k+1},\dots,\Lambda_g).$$

The order of vanishing: $N_k=|\Lambda^{[k]}|$.
The order of vanishings: n_k: The characteristics of the partition of $\Lambda^{[k]}$: $(a_1,\dots,a_{n_k};b_1,\dots,b_{n_k})$,
Riemann's singularity

Theorem 4.1. *If D_k belongs to $\mathcal{S}^k(X\backslash\infty)\setminus(\mathcal{S}^k_1(X)\cap\mathcal{S}^k(X\backslash\infty))$, and we let*

$$u:=\int_{k\infty}^{D_k}\nu^I,$$

1. *For every multi-index $(\alpha_1,\dots,\alpha_m)$ with $\alpha_i\in\{1,\dots,g\}$ and $m<n_k$,*

$$\frac{\partial^m}{\partial u_{\alpha_1}\dots\partial u_{\alpha_m}}\sigma(u)=0.$$

2. *There exists a multi-index $I_\beta:=(\beta_1,\dots,\beta_{n_k})$, which in general depends on D_k, such that*

$$\frac{\partial^{n_k}}{\partial u_{\beta_1}\dots\partial u_{\beta_{n_k}}}\sigma(u)\neq 0.$$

Corollary 4.2. *For $u^{[k]}\in\Theta^k\setminus\left(\Theta^k_1\cup\Theta^{k-1}\right)$, $u^{[g]}\in\mathbb{C}^g$, $v\in\mathcal{W}^1$, and $t\in\mathbb{R}$ $(0<|t|<1)$, and $\ell<N_k$, we have*

$$\left.\frac{\partial^\ell}{\partial {u^{[g]}_g}^{\ell}}\sigma(u^{[g]})\right|_{u^{[g]}=u^{[k]}}=0,\ \left.\frac{\partial^{N_k}}{\partial {u^{[g]}_g}^{N_k}}\sigma(u^{[g]})\right|_{u^{[g]}=u^{[k]}}\neq 0.$$

Proposition 4.3. *For the characteristics of the partition of $\Lambda^{[k]}$, $(a_1,a_2,\dots,a_{n_k};b_1,b_2,\dots,b_{n_k})$, the following holds:*

1. *There exists an integer ℓ_i such that*

$$\Lambda_{\ell_i}+g-\ell_i=a_i+b_i+1$$

for every $i=0,1,\dots,n_k$;

2. *When the correspondence is denoted by*

$$L^{[k]}(a_i,b_i):=\ell_i.$$

Definition 4.4. *Let* Index *be the family of all finite sequences made up with numbers between 1 and g. For an element I_k of* Index *and $u\in\mathbb{C}^g$, define:*

$$\sigma_{I_k}:=\left(\prod_{i\in I_k}\frac{\partial}{\partial u_i}\right)\sigma,$$

$$\deg_{w^{-1}}(I_k):=\sum_{i\in I_k}\deg_{w^{-1}}(u_i).$$

Definition 4.5. *For $k=1,2,\dots,g-1$, and the characteristics of the partition of $\Lambda^{[k]}$, $(a_1,\dots,a_r;b_1,\dots,b_r)$, we define*

$$\natural_k:=\{L^{[k]}(a_1,b_1),L^{[k]}(a_2,b_2),\dots,L^{[k]}(a_{n_k},b_{n_k})\},$$

and

$$\natural_k^{(i)}:=(\natural_k\setminus\{k+1\})\bigcup\{i\},\ \textit{for } i=1,2,\dots,k.$$

Further, $\natural_g:=\emptyset$ and $\natural_g^{(i)}:=\{i\}$ for $i=1,2,\dots,g$.

We continue the examples in Tables 2.1; for the case $(r,s)=(5,7)$ we have Table 5.1

Table 5.1 a

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(i)$	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	x^4	y^3	x^3y	x^2y^2
$N(i)$	0	5	7	10	12	14	15	17	19	20	21	22	24
Λ_i	-	12	8	7	5	4	3	3	2	1	1	1	1
Λ_i+g-i	-	23	18	16	13	11	9	8	6	4	3	2	1
n_i	4	4	3	3	3	2	2	1	1	1	1	1	-
N_i	48	36	28	21	16	12	9	6	4	3	2	1	-

Table 5.1 b

k	$(a_0,\dots,a_{n_k};b_0,\dots,b_{n_k})$	$(a_i+b_i+1)_{0\le i\le n_k}$	$\sum(a_i+b_i+1)$	$\natural_k$
0	(1, 4, 6, 11; 1, 4, 6, 11)	(3, 9, 13, 23)	48	(10, 6, 4, 1)
1	(0, 3, 5, 10; 0, 2, 5, 7)	(1, 6, 11, 18)	36	(12, 8, 5, 2)
2	(2, 4, 9; 1, 3, 6)	(4, 8, 16)	28	(9, 7, 3)
3	(1, 3, 8; 0, 2, 4)	(2, 6, 13)	21	(11, 8, 4)
4	(0, 2, 7; 0, 1, 3)	(1, 4, 11)	16	(12, 9, 5)
5	(1, 6; 1, 2)	(3, 9)	12	(10, 6)
6	(0, 5; 0, 2)	(1, 8)	9	(12, 7)
7	(4; 1)	(6)	6	(8)
8	(3; 0)	(4)	4	(9)
9	(2; 0)	(3)	3	(10)
10	(1; 0)	(2)	2	(11)
11	(0; 0)	(1)	1	(12)

Theorem 4.6. *Let $\mathcal{I}_g=\{\emptyset\}$. For each $k=1,2,\dots,g$, there exists a subfamily of* Index*, $\mathcal{I}_k$, of cardinality n_k, whose element I_k is such that $\deg_{w^{-1}}(I_k)\ge N_k$, and as a function over $\kappa^{-1}(\Theta^k\setminus(\Theta^k_1\cup\Theta^{k-1}))$,*

$$\sigma_{J_k}=\begin{cases}\neq 0 & \textit{for } J_k=I_k\\ =0 & \textit{for } J_k\subsetneq I_k.\end{cases}$$

Moreover, $\{\natural_k,\natural_k^{(k)},\natural_k^{(k-1)},\dots,\natural_k^{(2)},\natural_k^{(1)}\}\subset\mathcal{I}_k$.

Corollary 4.7. *For $u^{[k]} \in \kappa^{-1}(\mathcal{W}^k \setminus (\mathcal{W}_1^k \cup \mathcal{W}^{k-1}))$, the expansion of $\sigma_{\natural_k}(u^{[k]})$ at the origin takes the form*

$$\sigma_{\natural_k}(u^{[k]}) = S_{\Lambda^{(k)}}(T)|_{T_{\Lambda_i+g-i}=u_i^{[k]}} + \sum_{|\mathrm{w}_g(\alpha)|>|\Lambda|} c_\alpha^{[k]} \cdot (u^{[k]})^\alpha$$

where $c_\alpha \in \mathbb{Q}[\lambda_j]$ and $S_{\Lambda^{(a)}}(T)$ is the lowest-order term in the w-degree of the $u_i^{[k]}$; $\sigma_{\natural_k}(u^{[k]})$ is homogeneous of degree $|\Lambda^{(a)}|$ with respect to the λ-degrees.

The Jacobi inversion formula

Theorem 4.8. *For $k < g$, $(P_1, \dots, P_k) \in \mathcal{S}^k(X\backslash\infty) \setminus (\mathcal{S}_1^k(X) \cap \mathcal{S}^k(X\backslash\infty))$ and $u = \pm w(P_1, \dots, P_k) \in \kappa^{-1}(\Theta^k)$,*

$$\frac{\sigma_{\natural_k^{(i)}}(u)}{\sigma_{\natural_k}(u)} = (-1)^{k-i+1}\mu_{k,i-1}(P_1, \dots, P_k).$$

Proposition 4.9. *For $k < g$, $(P_1, \dots, P_k) \in \mathcal{S}^k(X\backslash\infty) \setminus (\mathcal{S}_1^k(X) \cap \mathcal{S}^k(X\backslash\infty))$, $u = \pm w(P_1, \dots, P_k) \in \kappa^{-1}(\Theta^k)$, subsequences $J_\ell = \left\{L^{[k]}_{(a_\ell,b_\ell)}, L^{[k]}_{(a_{\ell+1},b_{\ell+1})}, \dots, L^{[k]}_{(a_{n_k},b_{n_k})}\right\} \subset \natural_k$ ($\ell \le n_k + 1$) using the characteristics of the partition $(a_1, a_2, \dots, a_{n_k}; b_1, b_2, \dots, b_{n_k})$ of $\Lambda^{[k]}$, and $J_\ell^{(i)} := J_\ell \setminus \{k+1\} \cup \{i\}$ ($i = 1, 2, \dots, k$), the following relations hold:*

1. *For $\ell \le n_k$,*

$$\frac{\sigma_{J_\ell^{(i)}, g^{\deg_{w^{-1}}(\natural_k \setminus J_\ell)}}(u)}{\sigma_{J_\ell, g^{\deg_{w^{-1}}(\natural_k \setminus J_\ell)}}(u)} = (-1)^{k-i+1}\mu_{k,i-1}(P_1, \dots, P_k),$$

especially,

$$\frac{\sigma_{i+1, g^{N_k - \deg_{w^{-1}}(k+1)}}(u)}{\sigma_{k+1, g^{N_k - \deg_{w^{-1}}(k+1)}}(u)} = (-1)^{k-i+1}\mu_{k,i-1}(P_1, \dots, P_k).$$

2. *For $\ell \le n_k$, we have as a function over $\kappa^{-1}(\Theta^k \setminus (\Theta_1^k \cup \Theta^{k-1}))$,*

$$\sigma_{J_\ell^{(i)}, g^{\deg_{w^{-1}}(\natural_k \setminus J_\ell)}} \neq 0, \ \sigma_{J, g^{N'}} = 0,$$

where $0 \le N' \le \deg_{w^{-1}}(\natural_k \setminus J_\ell)$ and $J \subset J_\ell^{(i)}$ such that $\#J + N' < \#J_\ell^{(i)} + \deg_{w^{-1}}(\natural_k \setminus J_\ell)$, and $\ell_{\mathrm{a}} \in \Pi$,

$$\sigma_{J_\ell^{(i)}, g^{\deg_{w^{-1}}(\natural_k \setminus J_\ell)}}(u + \ell_{\mathrm{a}}) = \sigma_{J_\ell^{(i)}, g^{\deg_{w^{-1}}(\natural_k \setminus J_\ell)}}(u) \exp(L(u + \frac{1}{2}\ell_{\mathrm{a}}, \ell_{\mathrm{a}}))\chi(\ell_{\mathrm{a}}).$$

3. *For every $\ell = 1, 2, \dots, n_k$,*

$$\sigma_{J_\ell^{(i)}, g^{\deg_{w^{-1}}(\natural_k \setminus J_\ell)}}(u) = \epsilon_{k,J_\ell}\sigma_{\natural_k^{(i)}}(u), \ i = 1, \dots, k+1,$$

$$\sigma_{g^{N_k}}(u) = \epsilon_k \sigma_{\natural_k}(u),$$

where ϵ_{k,J_ℓ} and ϵ_k are non-vanishing rational numbers.

Proposition 4.10.

$$\varphi_n(P_1, P_2, \dots, P_n) := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}.$$

Let $u^((i) := w(P_i)$,

For $n < g$,

$$\frac{\sigma_{\natural_n^{(i)}}(\sum_i u^{(i)}) \prod_{i<j} \prod_{a=0}^{r-1} \sigma_{\natural_2}(u^{(i)} + \zeta_r^a u^{(j)})}{\prod_{i<j} \prod_{a=0}^{r-1} \sigma_{\natural_1}(u^{(i)})^{rn}}$$

$$= \epsilon\psi_n^{(i)}(P_1, P_2, \dots, P_n)\varphi_n^{r-1}(P_1, P_2, \dots, P_n)$$

For $n \ge g$

$$\frac{\sigma(\sum_i u^{(i)}) \prod_{i<j} \prod_{a=0}^{r-1} \sigma_{\natural_2}(u^{(i)} + \zeta_r^a u^{(j)})}{\prod_{i<j} \prod_{a=0}^{r-1} \sigma_{\natural_1}(u^{(i)})^{rn}}$$

$$= \epsilon\psi_n(P_1, P_2, \dots, P_n)\varphi_n^{r-1}(P_1, P_2, \dots, P_n)$$

prime form $(2, 2g+1)$ and $(3, 4)$ curve

$$\mathcal{E}(P, Q) = \frac{\sigma_{\natural_r}(u - v)}{\sqrt{du_1}\sqrt{du_2}}$$

which could be generalized as

$$\mathcal{E}(P, Q) = \frac{\sigma_{\natural_r g^K}(u - v)}{\sqrt{du_1}\sqrt{du_2}}$$

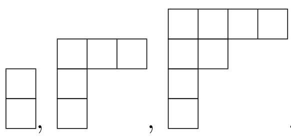
for a certain K.

addition formula $(2, 2g+1)$ and $(3, 4)$ curve

$$\frac{\sigma_{\natural_r}(u - v)\sigma_{\natural_2}(u + v)}{\sigma_{\natural_1}(u)^2\sigma_{\natural_1}(v)^2} = x - x'$$

5 A Curve (3,4,5)

The Young diagram: $\Lambda^{(3)} := (\Lambda_1^{(3)}, \Lambda_2^{(3)}, \Lambda_3^{(3)}) = (3, 1, 1)$ and $\Lambda^{(4)} := (\Lambda_1^{(4)}, \Lambda_2^{(4)}, \Lambda_3^{(4)}, \Lambda_4^{(4)}) = (4, 2, 1, 1)$:



The semigroup H_2 generated by $\langle 3, 4, 5 \rangle$

Two singular curves X_3 and X_4 generated by

$$f_{3,12}(x, y_4) := y_4^3 - k_4(x), \ k_4(x) := k_2(x)k_1(x)^2,$$
$$f_{4,15}(x, y_5) := y_5^3 - k_5(x), \ k_5(x) := k_2(x)^2k_1(x),$$

where for finite $b_a \in \mathbb{C}$ ($a = 1, 2, 3$) which is distinct from each other, and

$$k_2(x) := (x - b_1)(x - b_2) = x^2 + \lambda_1^{(2)}x + \lambda_2^{(2)},$$
$$k_1(x) := (x - b_0) = x + \lambda_1^{(1)}.$$

The commutative rings $R_3 := \mathbb{C}[x, y_4]/(f_{3,12}\ (x, y_4))$, $R_4 := \mathbb{C}[x, y_5]\ /(f_{4,15}(x, y_5))$.

$R_2 \equiv R := \mathbb{C}[x, y_4, y_5]/(f_8, f_9, f_{10})$, as a normalisation of these singular curves.

$$y_4y_5 = k_2(x)k_1(x), \ y_5 = \frac{y_4^2}{(x - b_0)}, \ y_4 = \frac{y_5^2}{(x - b_1)(x - b_2)}.$$

$f_8 = y_4^2 - y_5k_1(x)$, $f_9 = y_4y_5 - k_2(x)k_1(x)$, $f_{10} = y_5^2 - y_4k_2(x)$, as the 2×2 minors of

$$\begin{vmatrix} k_2(x) & y_4 & y_5 \\ y_4 & y_5 & k_3(x) \end{vmatrix}.$$

infinity point X is the Riemann surface which is naturally obtained as $X = X_2 \cup \{\infty\}$ as a set. with the unique infinity point ∞.
Gm action $\mathbb{G}_m$ acts on R by setting $g_m^{-3}x$, $g_m^{-a}y_a$ for x, y_a, $g_m \in \mathbb{G}_m$ and $a = 4, 5$.
the principal ideal theorem for X

$$y_4 = w_2 w_1^2,\ y_5 = w_2^2 w_1,$$

where

$$w_1^3 = k_1,\ w_2^3 = k_2.$$

$\tilde{R} := \mathbb{C}[x, w_1, w_2]/(w_1^3 - k_1(x), w_2^3 - k_2(x))$ over a natural covering of X.

5.1 The Weierstrass gap and holomorphic one forms

The Weierstrass gap sequences at ∞

$$x = \frac{1}{t_\infty^3},\ y_4 = \frac{1}{t_\infty^4}(1 + d_{\geq}(t_\infty)),\ y_5 = \frac{1}{t_\infty^5}(1 + d_{\geq}(t_\infty)),$$

The monomial curve $(3, 4, 5)$ is given by Pinkham

$$Z_4^2 = Z_3 Z_5,\ Z_4 Z_5 = Z_3^5,\ Z_5^2 = Z_3^3 Z_4,$$

$$\text{or the } 2 \times 2 \text{ minor of } \begin{vmatrix} Z_3 & Z_4 & Z_5 \\ Z_4 & Z_5 & Z_3^2 \end{vmatrix}.$$

Z_3, Z_4 and Z_5 correspond to $\dfrac{1}{x}$ $\dfrac{1}{y_4}$ and $\dfrac{1}{y_5}$ respectively.

Table 1

	0	1	2	3	4	5	6	7	8	9	10	11
X_3	1	-	-	x	y_4	-	x^2	xy_4	y_4^2	x^3	x^2y_4	xy_4^2
X_4	1	-	-	x	-	y_5	x^2	-	xy_5	x^3	y_5^2	x^2y_5
X_2	1	-	-	x	y_4	y_5	x^2	xy_4	xy_5	y_4y_5	x^2y_4	x^2y_5

There we define $\phi_i^{(g)}$ as a non-gap monomial in R_g for $g = 2, 3, 4$ and *e.g.*, $\phi_0^{(2)} = 1$, $\phi_1^{(2)} = x$, $\phi_2^{(2)} = y_4$, $\phi_3^{(2)} = y_5$, $\phi_4^{(2)} = x^2$, $\cdots$ and $\phi_0^{(3)} = 1$, $\phi_1^{(3)} = x$, $\phi_2^{(3)} = y_4$, $\phi_3^{(3)} = x^2$, $\phi_4^{(3)} = xy_4$, $\cdots$. We introduce the weight $N^{(g)}(n)$ by letting $N^{(g)}(n) := -\mathrm{wt}(\phi_n^{(g)})$.
Natural monomial $\phi_{H^1 i} \in R$ $(i = 1, 2, 3, \cdots)$:

$$\phi_{H^1 0} := y_4,\ \phi_{H^1 1} := y_5,\ \phi_{H^1 2} := xy_4,\ \phi_{H^1 3} := xy_5,$$

$$\text{for } i > 3,\ \phi_{H^1 i} := \begin{cases} x^{(i-4)/3} y_4 y_5 & i \equiv 1 \bmod 3, \\ x^{(i+1)/3} y_4 & i \equiv 2 \bmod 3,. \\ x^{i/3} y_5 & i \equiv 0 \bmod 3. \end{cases}$$

degree of singularity $N_{H^1}(n) := -\mathrm{wt}(\phi_{H^1 n})$; $N_{H^1}(0) = 4$, $N_{H^1}(1) = 5$, and $N_{H^1}(n) = n + 5$ for $n \geq 2$.

Proposition 5.1. *Bases of the holomorphic one forms over X can be expressed by*

$$\nu_1^I := \frac{dx}{3y_5},\ \nu_2^I := \frac{dx}{3y_4}.\ \text{ or } \nu_i^I := \frac{\phi_{H^1 i-1} dx}{3y_4 y_5},\ (i = 1, 2).$$

Proposition 5.2. $\sum_{i=0}^n a_i \tilde{\nu}_i$ *belongs to* $H^1(X \setminus \infty, \mathcal{O}_X)$, *where* $\tilde{\nu}_i := \frac{\phi_{H^1 i} dx}{3y_4 y_5}$ *and the order of the singularity of* $(\tilde{\nu}_i)$ *at* ∞ *is given by* $N_{H^1}(n) - 5$.

Lemma 5.3.

$$a_0 \frac{dx}{y_4 y_5} + a_1 \frac{xdx}{y_4 y_5} + a_2 \frac{x^2 dx}{y_4 y_5}$$

is not holomorphic one form over X if a_i does not vanish.

Homology bases α_i, β_j $(1 \leqq i, j \leqq 2)$ of $H_1(X, \mathbb{Z})$ such that their intersection numbers are $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{ij}$.
Period matrix

$$[\omega'\ \omega''] = \frac{1}{2} \left[\int_{\alpha_i} \nu_j^I \quad \int_{\beta_i} \nu_j^I \right]_{i,j=1,2}.$$

Lattice Π_2 be a lattice generated by ω' and ω''.
the abelian map $w : X \to \mathbb{C}^2$

$$w(P) = \int_\infty^P \nu^I \in \mathbb{C}^2,$$

and for a point $(P_1, \cdots, P_k) \in S^k X$, i.e., $w : S^k X \to \mathbb{C}^2$ by $w(P_1, \cdots, P_k) := \sum_{i=1}^k w(P_i)$.
the Jacobian $\mathcal{J}_2$ and its subvariety $\mathcal{W}_k$ $(k = 0, 1, 2)$ by

$$\kappa : \mathbb{C}^2 \to \mathcal{J}_2 = \mathbb{C}^2/\Pi_2 = \mathcal{W}_2,\ \mathcal{W}_k := \kappa w(S^k X)$$

respectively.

For a point $(P_1, P_2) \in S^2 X$ around the infinity point, by letting their local parameters $t_{\infty,1}$ and $t_{\infty,2}$ $u \equiv {}^t(u_1, u_2) := w(P_1, P_2)$ is given by

$$u_1 = \frac{1}{2}(t_{\infty,1}^2 + t_{\infty,2}^2)(1 + d_{>0}(t_{\infty,1}, t_{\infty,2})),$$
$$u_2 = (t_{\infty,1} + t_{\infty,2})(1 + d_{>0}(t_{\infty,1}, t_{\infty,2})),$$

where $d_{\geq}(t_1, t_2)$ is a natural extension of $d_{\geq}(t)$.

5.2 Differentials of the second and the third kinds

Following the EEL (Eilbeck-Enolskii-Leykin)-construction for a (n, s) curve,

Definition 5.4. *A two-form $\Omega(P_1, P_2)$ on $X \times X$ is called a fundamental differential of the second kind if it is symmetric,*

$$\Omega(P_1, P_2) = \Omega(P_2, P_1),$$

it has its only pole (of second order) along the diagonal of $X \times X$, and in the vicinity of each point (P_1, P_2) is expanded in power series as

$$\Omega(P_1, P_2) = \Big(\frac{1}{(t_{P_1} - t'_{P_2})^2} + d_{\geq}(1) \Big) dt_{P_1} \otimes dt_{P_2} \ \ (\text{as } P_1 \to P_2),$$

where t_P is a local coordinate at the point $P \in X$.

Proposition 5.5. *Let $\Sigma(P, Q)$ be the following form,*

$$\Sigma\big(P, Q\big) := \frac{y_{4,P} y_{5,P} + y_{4,P} y_{5,Q} + y_{4,Q} y_{5,P}}{(x_P - x_Q) 3 y_{4,P} y_{5,P}} dx_P.$$

Then $\Sigma(P, Q)$ has the properties

1. $\Sigma(P,Q)$ *as a function of* P *is singular at* $Q = (x_Q, y_{4,Q}, y_{5,Q})$ *and* ∞, *and vanishes at* $\hat{\zeta}_3^\ell(Q) = (x_Q, \zeta_3^\ell y_{4,Q}, \zeta_3^{2\ell} y_{5,Q})$, $(\ell = 1,2)$.

2. $\Sigma(P,Q)$ *as a function of* Q *is singular at* P *and at* ∞.

Proposition 5.6. *There exist differentials* $\nu_j^{II} = \nu_j^{II}(x, y_4, y_5)$ $(j = 1,2)$ *of the second kind such that they have their only pole at* ∞ *and satisfy the relation,*

$$\begin{aligned} &d_Q\Sigma\big(P,Q\big) - d_P\Sigma\big(Q,P\big) \\ &= \sum_{i=1}^{2}\Big(\nu_i^I(Q)\otimes\nu_i^{II}(P) - \nu_i^I(P)\otimes\nu_i^{II}(Q)\Big), \end{aligned}$$

where $d_Q\Sigma\big(P,Q\big)$ *given by*

$$dx_P \otimes dx_Q \frac{\partial}{\partial x_Q} \frac{y_{4,P}y_{5,P} + y_{4,P}y_{5,Q} + y_{4,Q}y_{5,P}}{(x_P - x_Q)3y_{4,P}y_{5,P}}.$$

The set of differentials $\{\nu_1^{II}, \nu_2^{II}\}$ *is determined modulo the* $\mathbb{C}$*-linear space spanned by* $\langle \nu_j^I \rangle_{j=1,2}$ *and the components of its representative one are*

$$\{\nu_1^{II}, \nu_2^{II}\} = \left\{ \frac{-\left(2x + \lambda_1^{(2)}\right)dx}{3y_4} . \frac{-xdx}{3y_5} \right\}.$$

We will fix this ν_i^{II} *hereafter.*

Corollary 5.7. 1. *The one form,*

$$\Pi_{P_1}^{P_2}(P) := \Sigma(P,P_1) - \Sigma(P,P_2),$$

is a differential of the third kind, whose only (first-order) poles are $P = P_1$ *and* $P = P_2$, *and residues* $+1$ *and* -1 *respectively.*

2. *The fundamental differential of the second kind* $\Omega(P_1,P_2)$ *is given by*

$$\begin{aligned} \Omega(P_1,P_2) &= d_{P_2}\Sigma(P_1,P_2) + \sum_{i=1}^{2}\nu_i^I(P_1)\otimes\nu_i^{II}(P_2) \\ &= \frac{F(P_1,P_2)dx_1\otimes dx_2}{(x_1-x_2)^2 9y_{4,P_1}y_{5,P_1}y_{4,P_2}y_{5,P_2}}, \end{aligned}$$

where F *is an element of* $R\otimes R$.

Lemma 5.8. *We have*

$$\lim_{P_1\to\infty}\frac{F(P_1,P_2)}{\phi_{H^1 1}(P_1)(x_1-x_2)^2} = \phi_{H^1 2}(P_2) = x_{P_2}y_{4,P_2}.$$

$$\begin{aligned} &\Omega_{Q_1,Q_2}^{P_1,P_2} := \int_{P_2}^{P_1}\int_{Q_2}^{Q_1}\Omega(P,Q) \\ &= \int_{P_2}^{P_1}(\Sigma(P,Q_1)-\Sigma(P,Q_2)) + \sum_{i=1}^{2}\int_{P_2}^{P_1}\nu_i^I(P)\int_{Q_2}^{Q_1}\nu_i^{II}(P). \end{aligned}$$

6 The sigma function over $(3,4,5)$ curve

6.1 Legendre relation

The complete integral of the second kind,

$$[\,\eta'\ \eta''] := \frac{1}{2}\left[\int_{\alpha_i}\nu_j^{II}\quad \int_{\beta_i}\nu_j^{II}\right]_{i,j=1,2}. \tag{6.2}$$

Let τ_{Q_1,Q_2} be the normalized differential of the third kind such that τ_{Q_1,Q_2} has residues $+1$, -1 at Q_1 and Q_2, is regular everywhere else, and is normalized, $\int_{\alpha_i}\tau_{P,Q} = 0$ for $i = 1,2$.

Lemma 6.1. *By letting* $\gamma = {\omega'}^{-1}\eta'$, *we have*

$$\Omega_{Q_1,Q_2}^{P_1,P_2} = \int_{P_2}^{P_1}\tau_{Q_1,Q_2} + \sum_{i,j=1}^{2}\gamma_{ij}\int_{P_2}^{P_1}\nu_i^I\int_{Q_2}^{Q_1}\nu_j^I.$$

generalized Legendre relation

Proposition 6.2. (generalized Legendre relation) *The matrix,*

$$M := \begin{bmatrix} 2\omega' & 2\omega'' \\ 2\eta' & 2\eta'' \end{bmatrix}, \tag{6.3}$$

satisfies

$$M\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}{}^tM = 2\pi\sqrt{-1}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}. \tag{6.4}$$

Proposition 6.3. *For* $(P,P_1,P_2) \in X\times S^2(X)\setminus S_1^2(X)$ *and* $u := w(P_1,P_2)$, *the equality*

$$\sum_{i,j=1}^{2}\wp_{i,j}\left(w(P)-u\right)\phi_{H^1 i-1}(P)\phi_{H^1 j-1}(P_a) = \frac{F(P,P_a)}{(x-x_a)^2},$$

holds for every $a = 1,2$, *where we set*

$$\wp_{ij}(u) := -\frac{\sigma_i(u)\sigma_j(u)-\sigma(u)\sigma_{ij}(u)}{\sigma(u)^2} \equiv -\frac{\partial^2}{\partial u_i\partial u_j}\log\sigma(u).$$

6.2 The μ_n functions over $(3,4,5)$

Frobenius-Stickelberger (FS) matrix and determinant:

$$\Psi_n(P_1,P_2,\ldots,P_n) := \begin{pmatrix} \phi_{H^1 0}(P_1) & \phi_{H^1 1}(P_1) & \cdots & \phi_{H^1 n-1}(P_1) \\ \phi_{H^1 0}(P_2) & \phi_{H^1 1}(P_2) & \cdots & \phi_{H^1 n-1}(P_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{H^1 0}(P_n) & \phi_{H^1 1}(P_n) & \cdots & \phi_{H^1 n-1}(P_n) \end{pmatrix}.$$

$\psi_n(P_1,P_2,\ldots,P_n) := \det\Psi_n(P_1,P_2,\ldots,P_n)$.

Definition 6.4. *For* $P,P_1,\ldots,P_n \in (X\backslash\infty)\times\mathcal{S}^n(X\backslash\infty)$, *we define* $\mu_n(P)$ *by*

$$\begin{aligned} \mu_n(P) &:= \mu_n(P;P_1,\ldots,P_n) \\ &:= \lim_{P_i'\to P_i}\frac{1}{\psi_n(P_1',\ldots,P_n')}\psi_{n+1}(P_1',\ldots,P_n',P), \end{aligned}$$

where the P_i' *are generic, the limit is taken (irrespective of the order) for each* i; *and* $\mu_{n,k}(P_1,\ldots,P_n)$ *by*

$$\mu_n(P) = \phi_{H^1 n}(P) + \sum_{k=0}^{n-1}(-1)^{n-k}\mu_{n,k}(P_1,\ldots,P_n)\phi_{H^1 k}(P).$$

$$\mu_2(P;P_1,P_2) = xy_4 - \frac{y_{4,1}x_2y_{4,2} - y_{4,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}}y_5 + \frac{y_{5,1}x_2y_{4,2} - y_{5,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}}y_4,$$

$$\mu_1(P;P_1) = y_5 + \frac{y_{5,1}}{y_{4,1}}y_4.$$

We note that $\mu_{H^1,n}$ for X is characterized by the condition on a polynomial $\alpha_{H^1 n} = \sum_{i=0}^n a_i \phi_{H^1 i}(P)$, $a_i \in \mathbb{C}$ and $a_n = 1$, which has a zero at each point P_i and has the smallest possible order such that it multiplied by $dx/3y_4y_5$ belongs to $H^1(X \setminus \infty, \mathcal{O}_X)$.

6.3 Jacobi inversion formulae over Θ^k

Theorem 6.5. (Jacobi inversion formula) *For $(P, P_1, P_2) \in X \times \big(S^2(X) \setminus S^2_1(X)\big)$, we have*

1. $\mu_{H^1,2}(P;P_1,P_2) = xy_4 - \wp_{22}(w(P_1,P_2))y_4 + \wp_{21}(w(P_1,P_2))y_5$.
2. $\wp_{22}(w(P_1,P_2)) = \dfrac{y_{4,1}x_2y_{4,2} - y_{4,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}}$,
   $\wp_{21}(w(P_1,P_2)) = \dfrac{y_{5,1}x_2y_{4,2} - y_{5,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}}$.

Theorem 6.6. *For $\Theta^1 := \mathcal{W}^1 = [-1]\mathcal{W}^1$, and $P_1 \in X \setminus S^1_1(X)$ and $u = \pm w(P_1) \in \kappa^{-1}(\Theta^1)$,*

$$\frac{\sigma_1(u)}{\sigma_2(u)} = \frac{y_5}{y_4}$$

$$\sigma(u) = \frac{1}{2}u_2^2 - u_1 + \sum_{|\alpha|>2} a_\alpha u^\alpha,$$

where $a_\alpha \in \mathbb{Q}[b_1, \cdots, b_5]$, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$ and $u^\alpha = u_1^{\alpha_1} u_2^{\alpha_2}$. Since for a Young diagram Λ, S_Λ and s_Λ are the Schur functions defined by

$$S_\Lambda(T_1, T_2) = s_\Lambda(t_1, t_2) = t_1 t_2 = \frac{1}{2}T_1^2 - T_2,$$

where $T_1 := t_1 + t_2$ and $T_2 := \frac{1}{2}(t_1^2 + t_2^2)$, we have

$$\sigma(u) = S_\Lambda(u_1, u_2) + \sum_{|\alpha|>2} a_\alpha u^\alpha.$$