On Submanifold Dirac Operators and Generalized Weierstrass Relations

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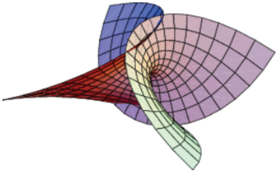
Purpose of Study of Submanifold Dirac Operator

It is known that the Dirac operator associated with a principal bundle provides the data of the principal bundle, via, Atiyha-Singer index theorem.
$\rightarrow$ To find a Dirac operator which provides the data of immersion of submanifolds in a manifold, e.g, $\mathbb{E}^n$, and to investigate its properties.

Weierstrass Relation of Minimal Surface

Weierstrass Relation

Evey minimal surface $X : S \hookrightarrow \mathbb{E}^3$ is given by a solution of the differential equation. The moduli of minimal sufaces completely agree with the moduli of the solutions.



Minimal Surface
$X : S \hookrightarrow \mathbb{E}^3$
such that mean
curvature vanishes

$$\begin{pmatrix} & \partial_z \\ \overline{\partial_z} & \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0$$

$$\Leftrightarrow$$

$$X^1 + \sqrt{-1}X^2 = \sqrt{-1} \int ((\bar{g})^2 dz - (\bar{f})^2 d\bar{z})$$

$$X^1 - \sqrt{-1}X^2 := \overline{X^1 + \sqrt{-1}X^2}$$

$$X^3 = -\sqrt{-1} \int ((\bar{g} f) dz + (\bar{f} g) d\bar{z})$$

Submanifold Dirac Operator

Today's topic on Submanifold Dirac Operator

Assume that a smooth manifold $\mathcal{M}$ is smoothly embedded in $\mathbb{E}^n$. The solution space of the submanifold Dirac equation $D_{\mathcal{M}\hookrightarrow\mathbb{E}^n}\psi = 0$ **locally** provides the embedding data $\mathcal{M} \hookrightarrow \mathbb{E}^n$, i.e., a generalization of the Weierstrass relation for the minimal surface.
(M. 2008 Adv. Study Pure Math.)

Open Problems

However its **global properties** are open problems.

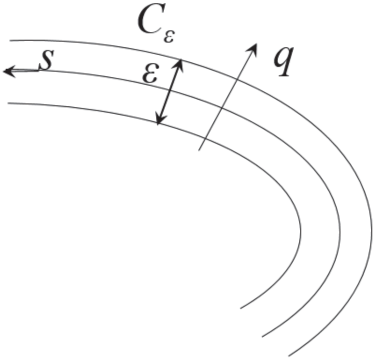
Submanifold Quantum Mechanics

History of Submanifold Quantum Mechanics

1. 1971: H. Jensen and H. Koppe: *Quantum Mechanics with Constraints* Ann. Phys.
2. 1981: R. C. T. da Costa *Quantum mechanics of a constrained particle* Phys. Rev. A
3. 1989-1995 Pavel Exner and P. Šeba, M. Ikegami, Y. Nagaoka, S. Takagi and T. Tanzawa, M-Tsuru, and so on: shape effect of the quantum devices e.g., fine structures of semiconductor devices
4. 1992-2008: Dirac operator, M-Tsuru, M, Burgress-Jensen
5. 1989-2000: It is the same as operators coming from differential geometrical relations, e.g., the Gauss-Codatzi equation, so on. Kenmotsu, Pedit-Pinkall, Bobenko, Konopelchenko, Taimanov etc, which is regarded as a generalization of Weierstrass equation.
6. 2000-2008: M: Mathematical construction (in terms of algebraic analysis.)
7. 2010-2011: de Oliveira: Gamma-convergence of this problem (Direchlet tube) of Laplacian is studied.

Submanifold Quantum Mechanics

A Tubular neighborhood C_ε of a curve C in the two-euclidean space $\mathbb{E}^2$



$C := \lim_{\varepsilon \to 0} \overline{C_\varepsilon}$

s : **arclength of** C

q : **normal coordinate**

k : **curvature of** C

$k(s) = 1/R(s)$

R : **curvature radius**

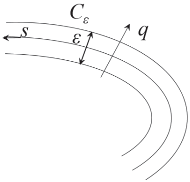
Submanifold Quantum Mechanics

To describe the Schrödinger particles confined in C_ε after taking a limit $\varepsilon \to 0$,

Submanifold Quantum Mechanics

$$(-\Delta + V_\varepsilon(x))\Psi = E\Psi$$

where $V_\varepsilon(x) = \begin{cases} 0 & \textbf{for } x \in C_\varepsilon \\ \infty & \textbf{otherwise} \end{cases}$



Problem: assuming $\Psi = \psi(s)\varphi(q)$ for $\varepsilon \to 0$, to find an equation which governs $\psi(s)$ for $\varepsilon \to 0$!

Answer: $\left(-\frac{\partial^2}{\partial s^2} - \frac{1}{4}k^2\right)\psi(s) = E\psi(s) \qquad k = k(s)$**: curvature**

Submanifold Quantum Mechanics

pre-Hilbert space

Consider the pre-Hilbert space $\mathcal{H} := (\Omega, \langle\ ,\ \rangle)$ for $\Omega := \mathcal{C}^\infty(\mathbb{E}^2, \mathbb{C})$, the bijection $\alpha : \Omega \to \Omega^* = \mathcal{C}^\infty(\mathbb{E}^2, \mathbb{C})$
with the inner product

$$\langle \psi, \varphi \rangle := \int \overline{\psi(x)} \varphi(x) dx^1 dx^2,$$

$\psi, \varphi \in \Omega := \mathcal{C}^\infty(\mathbb{E}^2, \mathbb{C})$,
x^1, x^2: Cartesian coordinate

Due to the potential V_ε, $\mathrm{supp}\psi \subset \overline{C_\varepsilon}$. Let us express $\mathcal{H}$ in terms of curved coordinate $(\xi^1, \xi^2) = (s, q)$.

Submanifold Quantum Mechanics

Curved coordinate

The coordinate transformation $x = (x^1, x^2) \to \xi = (\xi^1, \xi^2) = (s, q)$, $d\ell^2 = \delta_{ij} dx^i dx^j = \eta_{ij} d\xi^i d\xi^j$ induces the following:

1. The Riemannian metric: $(\eta_{ij}) = \begin{pmatrix} (1+k(s)q)^2 & \\ & 1 \end{pmatrix}$,
   $d\ell^2 = (1+kq)^2 ds^2 + dq^2$, $\quad (\eta^{ij}) = (\eta_{ij})^{-1}$.
2. $\eta = \det(\eta_{ij}) = (1+k(s)q)^2$, with volume form $\eta^{1/2} dsdq$
3. Inner product is $\langle \psi, \varphi \rangle = \int \overline{\psi(\xi)} \varphi(\xi) \eta^{1/2} dsdq$
4. $\Delta = \eta^{-1/2} \partial_{\xi_i} \eta^{1/2} \eta^{ij} \partial_{\xi_j}$: Beltrami-Laplace operator
   $$= \frac{1}{1+kq} \left(\frac{\partial}{\partial s} \frac{1}{1+kq} \frac{\partial}{\partial s} + \frac{\partial}{\partial q} (1+kq) \frac{\partial}{\partial q} \right)$$

Key Lemma

In $\mathcal{H} := (\Omega, \langle\ ,\ \rangle)$, $\sqrt{-1}\partial_q$ is not self-adjoint!

proof:

Noting $\eta^{1/2} = (1 + kq)$ and $\mathrm{supp}(\psi) \subset \overline{C_\varepsilon}$, the partial integral leads

$$\langle\psi, \sqrt{-1}\partial_q\varphi\rangle = \int \overline{\psi(\xi)}(\sqrt{-1}\partial_q\varphi(\xi))\eta^{1/2}dsdq$$
$$= \left\langle \left(\sqrt{-1}\partial_q - \frac{k(s)}{1 + k(s)q}\right)\psi, \varphi \right\rangle.$$

We thus have $= \left(\sqrt{-1}\partial_q\right)^* = \left(\sqrt{-1}\partial_q - \frac{k(s)}{1 + k(s)q}\right).$

Physically speaking

Physically $\sqrt{-1}\partial_q$ is not an observable and thus its eigenvalue is not real. ($\rightarrow$ **We cannot confine it by $\sqrt{-1}\partial_q$ in $\mathcal{H}$.**)

Submanifold Quantum Mechanics

Hörmander's half-density form

Apply the half-density form to the problem: $\tilde{\psi} = \eta^{1/4}\psi$ and $\langle\tilde{\psi},\tilde{\varphi}\rangle_\eta = \int \overline{\tilde{\psi}(\xi)}\tilde{\varphi}(\xi) dsdq := \langle\psi,\varphi\rangle$. (Jacobian $\eta^{1/2}$ disappears!)

Deformation of pre-Hilbert space,

$$\tau : \mathcal{H} = (\Omega, \langle\circ,\bullet\rangle) \to \tilde{\mathcal{H}} = (\tilde{\Omega}, \langle\circ,\bullet\rangle_\eta)$$

$$\left(\psi, \int(\overline{\circ},\bullet)\eta^{1/2} dsdq\right) \mapsto \left(\eta^{1/4}\psi, \int(\overline{\circ},\bullet) dsdq\right),$$

an operator P: $\tau : P \mapsto \eta^{1/4}P\eta^{-1/4}$ such that

$$\langle\psi, P\varphi\rangle = \langle\tilde{\psi}, \eta^{1/4}P\eta^{-1/4}\tilde{\varphi}\rangle.$$

Submanifold Quantum Mechanics

Lemma 1: Self-adjointness of the normal operators

In the deformed Hilbert space, $\tilde{\mathcal{H}} = (\tilde{\Omega}, \langle\ ,\ \rangle_\eta)$,
$\sqrt{-1}\partial_q$ is self-adjoint, i.e., $\langle\tilde{\psi}, \sqrt{-1}\partial_q\tilde{\varphi}\rangle_\eta = \langle\sqrt{-1}\partial_q\tilde{\psi}, \tilde{\varphi}\rangle_\eta$

proof:

Noting no Jacobian $\eta^{1/2} = (1 + kq)$ in the volume form and $\mathrm{supp}(\psi) \subset \overline{C_\varepsilon}$, the partial integral leads

$$\begin{aligned}\langle\tilde{\psi}, \sqrt{-1}\partial_q\tilde{\varphi}\rangle_\eta &= \int \overline{\tilde{\psi}(\xi)}(\sqrt{-1}\partial_q\tilde{\varphi}(\xi))dsdq \\ &= \int \overline{(\sqrt{-1}\partial_q\tilde{\psi}(\xi))}\tilde{\varphi}(\xi)dsdq\end{aligned}$$

Physically speaking

Physically $\sqrt{-1}\partial_q$ in $\tilde{\mathcal{H}}$ is an observable and thus its eigenvalue is real.
($\rightarrow$ **We can confine it by $\sqrt{-1}\partial_q$ in $\tilde{\mathcal{H}}$.**)

Submanifold Quantum Mechanics

Submanifold Quantum Mechanics

By the deformation, we have the Schrödinger equation

$$H := (-\Delta + V_\varepsilon) \to \eta^{1/4} H \eta^{-1/4} \quad \text{in} \quad \tilde{\mathcal{H}}$$

$$\left(-\frac{1}{1+kq}\frac{\partial}{\partial s}\frac{1}{1+kq}\frac{\partial}{\partial s} - \frac{1}{4}\frac{k^2}{1+kq} - \frac{\partial^2}{\partial q^2} + V_\varepsilon + o(q)\right)\tilde{\psi} = E\tilde{\psi}$$

and apply the separation of variables method to this, $\tilde{\psi}(\xi) = \psi_\parallel(s)\psi_\perp(q)$, under $\varepsilon \to 0$ $(q \to 0)$ to obtain

$$\text{parallel} \ : \ \left(-\frac{\partial^2}{\partial s^2} - \frac{1}{4}k^2(s)\right)\psi_\parallel = E_\parallel \psi_\parallel$$

$$\text{normal} \ : \ \left(-\frac{\partial^2}{\partial q^2} + V_\varepsilon + o(q)\right)\psi_\perp = E_\perp \psi_\perp, \quad (E_\perp \to \infty, (\varepsilon \to 0)).$$

Submanifold Quantum Mechanics

The problem of Submanifold Quantum Mechanics

To describe the Schrödinger particles confined in C_ε after taking a limit $\varepsilon \to 0$,

Submanifold Quantum Mechanics (Jensen-Koppe, da Costa)

The submanifold Schrödinger equation describing the Schrödinger particles confined in C_ε of $\varepsilon \to 0$ is $\left(-\frac{\partial^2}{\partial s^2} - \frac{1}{4}k^2\right)\psi_\parallel = E_\parallel \psi_\parallel$.

Its more algebraic construction

We want to find more algebraic construction because the normal behavior diverges in general ($E_\perp \to \infty$ for $\varepsilon \to 0$).
Key fact is that **for the parallel equation, both ∂_q and q behave like zero after deformation of the pre-Hilbert space by the half-density form**.

Algebraic Construction

Algebraic Construction of Submanifold Quantum Mechanics

The submanifold Schrödinger operator is algebraically obtained: (M 2003)

$$\left(-\frac{\partial^2}{\partial s^2} - \frac{1}{4}k^2\right) = -\eta^{1/4}\Delta\eta^{-1/4}\Big|_{\mathrm{Ker}\partial_q, q=0}$$
$$= -\eta^{-1/4}\partial_{\xi_i}\eta^{1/2}\eta^{ij}\partial_{\xi_j}\eta^{-1/4}\Big|_{\mathrm{Ker}\partial_q, q=0}$$
$$= \left(-\frac{1}{1+kq}\frac{\partial}{\partial s}\frac{1}{1+kq}\frac{\partial}{\partial s} - \frac{1}{4}\frac{k^2}{1+kq} - \frac{\partial^2}{\partial q^2} + o(q)\right)\Big|_{\mathrm{Ker}\partial_q, q=0}$$

Here instead of the potential V_ε, we restrict the domains

$\tilde{\Omega} := \mathrm{dom}(\eta^{1/4}\Delta\eta^{-1/4})$ to $\mathrm{Ker}\partial_q$

$\tilde{\Omega}^*$ to $\mathrm{Ker}(\partial_q)^*$

and put $q = 0$.

Why the deformation $\mathcal{H} \to \tilde{\mathcal{H}}$ is needed

Since $(\sqrt{-1}\partial_q)^* = \sqrt{-1}\partial_q$ in $\tilde{\mathcal{H}}$, we have that $(\mathrm{Ker}\partial_q)^* = \mathrm{Ker}(\partial_q^*)$.

Lemma: (Well-definedness of projection)

$$\pi : \tilde{\Omega} \to \mathrm{Ker}\partial_q, \quad \pi^* : \tilde{\Omega}^* \to \mathrm{Ker}(\partial_q^*)$$

where $\tilde{\Omega} := \mathcal{C}^\infty(C_\varepsilon, \mathbb{C})$, $\alpha : \tilde{\Omega} \to \tilde{\Omega}^*$ $(\psi \mapsto \overline{\psi})$

We have the commutative diagram

$$\begin{array}{ccc} \tilde{\Omega} & \xrightarrow{\alpha} & \tilde{\Omega}^* \\ \downarrow \pi & & \downarrow \pi^* \\ \mathrm{Ker}\partial_q & \xrightarrow{\alpha} & (\mathrm{Ker}\partial_q)^* \end{array},$$

and $\alpha|_{\mathrm{Ker}\partial_q}$ is bijection. $(\mathrm{Ker}\partial_q, \langle , \rangle)$ becomes the pre-Hilbert space.

Algebraic Construction of Submanifold Quantum Mechanics (2003 M)

$$\left(-\frac{\partial^2}{\partial s^2} - \frac{1}{4}k^2\right) = -\pi^* \left(\eta^{1/4}\Delta\eta^{-1/4}\right) \pi\Big|_{q=0}$$

Summary of Submanifold Quantum Mechanics

Summary of Submanifold Quantum Mechanics

1. Consider a differential operator Q on the euclidean space $\mathbb{E}^n$, e.g., $Q = -\Delta$, $Q = \sqrt{-1}\,\not{D}$
2. Consider a n-tubular neighborhood $\mathcal{M}_\varepsilon \subset \mathbb{E}^n$ parameterized by ε of a k-Riemannian (Spin) submanifold $\mathcal{M} = \mathcal{M}_\varepsilon|_{\varepsilon=0}$, $(\mathcal{M}, \eta_{\|\alpha\beta})$. $(\mathcal{M} \subset \mathcal{M}_\varepsilon \subset \mathbb{E}^n)$
3. Let the local parameter of $\mathcal{M}$ be s and that of normal direction be q so that the metric η is decomposed like $(\eta_{ij}) = \eta_{\|\alpha\beta} \oplus \eta_{\perp\mu\nu}$ where $\eta_\perp$ is normal one $\eta_{\perp\mu\nu}(pt) = \delta_{\mu\nu}$ $pt \in \mathcal{M}_\varepsilon$.
4. Deform $Q_\eta := \eta_\|^{1/4} Q \eta_\|^{-1/4}$ by the half-density form so that $\sqrt{-1}\partial_{q^\mu}$ $(\mu = k+1, \dots, n)$ along the normal direction is self-adjoint.
5. Restrict the domain of Q_η to $\bigcap_{\mu=k+1}^n \mathrm{Ker}(\partial_{q^\mu})$, and put $q = 0$:

$$Q_{\mathcal{M}\hookrightarrow\mathbb{E}^n} := Q_\eta|_{\bigcap_{\mu=k+1}^n \mathrm{Ker}(\partial_{q^\mu}), q=0}$$

Submanifold Schrödinger Operator

Submanifold Schrodinger operator

$n = 3$, $k = 2$ case: A surface S in $\mathbb{E}^3$ case: (da Costa 1981)

$$-\Delta_{S \hookrightarrow \mathbb{E}^2} = -\Delta_S - \frac{1}{4}(K_c - K_m^2),$$

where
Δ_S is Beltrami-Laplacian of S,
K_c is Gauss curvature of S, and
K_m is mean curvature of $S \hookrightarrow \mathbb{E}^3$,

Submanifold Dirac Operator

On Submanifold Dirac Operator

Construction of Submanifold Dirac Operator

1. Consider the Dirac operator $\not{D}_{\mathbb{E}^n}$ in $\mathbb{E}^n$.
2. Consider a spin submanifold $\mathcal{M} \hookrightarrow \mathbb{E}^n$.
3. Consider its tubular neighborhood $\mathcal{M}_\varepsilon \hookrightarrow \mathbb{E}^n$ and define q^μ $(\mu = k+1, \dots, n)$ in normal direction.
4. Apply the half-density form so that $\sqrt{-1}\partial_{q^\mu}$ $(\mu = k+1, \dots, n)$ is self-adjoint. and deform $\not{D}_{\mathbb{E}^n}$ to obtain $\eta_{\|}^{1/4} \not{D}_{\mathbb{E}^n} \eta_{\|}^{-1/4}$ by the half-density form
5. Restrict its domain to $\bigcap_{\mu=k+1}^n \mathrm{Ker}(\partial_{q^\mu})$ and put $q = 0$.

$$\not{D}_{\mathcal{M} \hookrightarrow \mathbb{E}^n} := \eta_{\|}^{1/4} \not{D}_{\mathbb{E}^n} \eta_{\|}^{-1/4} \Big|_{\bigcap_{\mu=k+1}^n \mathrm{Ker}(\partial_{q^\mu}), q=0}$$

Submanifold Dirac Operator

Submanifold Dirac Operator and Geometrical Data

1. For $k < n$, the embedding $\iota_{k,n} : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$ can be expressed in terms of the dual space $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$, $\langle e_j^*, \iota_{k,n}(b_\alpha)\rangle$.
2. The Clifford module $\mathrm{Cliff}(\mathbb{E}^n) \approx \mathbb{C}^{2^{[n/2]}}$ naturally contains a dual space $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$ and its bases e_j^*.
3. $\mathrm{Ker}\, \not{D}_{\mathbb{E}^n} \subset \mathcal{C}^\infty(\mathbb{E}^n, \mathbb{C}^{2^{[n/2]}})$ reproduces the Clifford module $\mathrm{Cliff}(\mathbb{E}^n) \approx \mathbb{C}^{2^{[n/2]}}$.
4. The kernel of the submanifold Dirac operator $\not{D}_{\mathcal{M}\hookrightarrow\mathbb{E}^n}$ also reproduces $\mathrm{Cliff}(\mathbb{E}^n)|_{\mathcal{M}}$. For every $p \in \mathcal{M} \subset \mathbb{E}^n$ the kernel provides the embedding data $\iota_{k,n} : T_p\mathcal{M} \hookrightarrow T_p\mathbb{E}^n$, or $\langle e_j^*, \iota_{k,n}(b_\alpha)\rangle$.
   This is a kind of the generalized Weierstrass relation.

Brief Review of Clifford Algebra

Clifford Group

Clifford Algebra

Clifford Algebra $\mathrm{CLIFF}(\mathbb{R}^n) := \mathbb{T}(\mathbb{R}^n)/((v, u)_{\mathbb{R}^n} - 1)$,
where $(v, u)_{\mathbb{R}^n}$ is the euclidean inner product $\mathbb{R}^n$.

Graded Algebra in Clifford Algebra

$$\mathrm{CLIFF}(\mathbb{R}^n) := \bigoplus_{p=0}^{n} \mathrm{CLIFF}^p(\mathbb{R}^n).$$

$$\mathrm{CLIFF}^{\mathrm{even}}(\mathbb{R}^n) := \bigoplus_{p=\mathrm{even}}^{n} \mathrm{CLIFF}^p(\mathbb{R}^n).$$

γ- matrices in Clifford Algebra & *-Operator

$$\gamma : \mathbb{R}^n \to \mathrm{CLIFF}^1(\mathbb{R}^n), \quad (e^i \mapsto \gamma(e^i))$$
$$(\gamma(e^{i_1}) \cdots \gamma(e^{i_j}))^* := (\gamma(e^{i_j}) \cdots \gamma(e^{i_1})).$$

Irreducible Clifford Algebra

Irreducible Left Representation $\text{Cliff}(\mathbb{R}^n)$

$\text{Cliff}(\mathbb{R}^n) \approx \mathbb{C}^{2^{[n/2]}}$, and $\beta : \text{CLIFF}^{\mathbb{C}}(\mathbb{R}^n) \to \text{END}(\text{Cliff}(\mathbb{R}^n))$

Irreducible Right Representation $\text{Cliff}^*(\mathbb{R}^n)$

$\text{Cliff}(\mathbb{R}^n) \approx \mathbb{C}^{2^{[n/2]}*}$, and $\beta^* : \text{CLIFF}^{\mathbb{C}}(\mathbb{R}^n) \to \text{END}(\text{Cliff}^*(\mathbb{R}^n))$

$\text{Cliff}(\mathbb{R}^n)$ and $\text{Cliff}^*(\mathbb{R}^n)$

1. $\alpha : \text{Cliff}(\mathbb{R}^n) \to \text{Cliff}(\mathbb{R}^n)^*$.
   For $C \in \text{CLIFF}(\mathbb{R}^n)$ and $c \in \text{Cliff}(\mathbb{R}^n)$, $\alpha(Cc) = \alpha(c)C^*$
   $\overline{c} := \alpha(c)$.
2. Pairing: $\langle\rangle_{\text{Cliff}(\mathbb{R}^n)} : \text{Cliff}(\mathbb{R}^n)^* \times \text{Cliff}(\mathbb{R}^n) \to \mathbb{C}$:

Clifford Group

Definition of Clifford Group

By letting $\mathrm{CLIFF}^{\mathrm{even},\times}(\mathbb{R}^n)$ be a multiple subset of $\mathrm{CLIFF}^{\mathrm{even}}(\mathbb{R}^n)$, the Clifford group $\mathrm{CG}(\mathbb{R}^n)$ is defined by

$$\{\tau \in \mathrm{CLIFF}^{even\times}(\mathbb{R}^n) \mid \text{ for } \forall v \in \mathrm{CLIFF}^1(\mathbb{R}^n), \tau v \tau^* \in \mathrm{CLIFF}^1(\mathbb{R}^n)\}.$$

Lemma: Properties of Clifford Group $\mathrm{CG}(\mathbb{R}^n)$

1. $\tau \in \mathrm{CG}(\mathbb{R}^n)$, $\tau : \mathrm{CLIFF}^1(\mathbb{R}^n) \to \mathrm{CLIFF}^1(\mathbb{R}^n)$, $(\tau(v) = \tau v \tau^*)$
2. Group homomorphism: $\mathrm{CG}(\mathbb{R}^n) \to \mathrm{GL}(n, \mathbb{R})$
3. Representation of $\mathrm{CG}(\mathbb{R}^n)$, via $c \mapsto \tau c$ for $c \in \mathrm{Cliff}(\mathbb{R}^n) \approx \mathbb{C}^{2^{[n/2]}}$ and $\tau \in \mathrm{CG}(\mathbb{R}^n) \subset \mathrm{GL}(2^{[n/2]}, \mathbb{C})$.

Natural Embedding

Key Lemma:

1. For $c \in \mathrm{Cliff}(\mathbb{R}^n)$,

$$\langle \overline{c}, \gamma(\circ)c\rangle_{\mathrm{Cliff}(\mathbb{R}^n)} : \mathbb{R}^n \to \mathbb{C} \quad (v \mapsto \langle \overline{c}, \gamma(v)c\rangle_{\mathrm{Cliff}(\mathbb{R}^n)})$$

   is a $\mathbb{R}$-linear map.
2. There is a subspace $\mathrm{Cliff}^{pr}(\mathbb{R}^n) \subset \mathrm{Cliff}(\mathbb{R}^n)$ such that for $c \in \mathrm{Cliff}^{pr}(\mathbb{R}^n)$,

$$\langle \overline{c}, \gamma(\circ)c\rangle_{\mathrm{Cliff}(\mathbb{R}^n)} : \mathbb{R}^n \to \mathbb{R} \quad (v \mapsto \langle \overline{c}, \gamma(v)c\rangle_{\mathrm{Cliff}(\mathbb{R}^n)} \in \mathbb{R})$$

   is a $\mathbb{R}$ linear map.
3. $\mathrm{Cliff}^{pr}(\mathbb{R}^n)$ is bijective to $(\mathbb{R}^n)^*$.
4. $\mathrm{Cliff}^{pr}(\mathbb{R}^n)$ reproduces the map $\iota : \mathbb{R}^k \hookrightarrow \mathbb{R}^m$, $\iota \in \dfrac{\mathrm{GL}(n,\mathbb{R})}{\mathrm{GL}(n-k,\mathbb{R})\mathrm{GL}(k,\mathbb{R})}$, i.e., $\langle e_i^*, \iota(b_\alpha)\rangle$

Submanifold Dirac operator $\not{D}_{S \hookrightarrow \mathbb{E}^4}$
of a Conformal Surface S Embedded in $\mathbb{E}^4$

Clifford Module of $\mathbb{E}^4$

1. γ-matrices for the bases $\{dx^i\}$ of $\mathbb{R}^4 = T^p\mathbb{E}^4$
   $\gamma_{\{dx\}}(dx^i) = \sigma^1 \otimes \sigma^i$ for $i = 1, 2, 3$ and $\gamma_{\{dx\}}(dx^4) = \sigma^2 \otimes 1$.
2. By letting $\mathbb{E}^4 \approx \mathbb{C} \times \mathbb{C} \ni (Z^1, Z^2) \equiv (x^1 + \sqrt{-1}x^2, x^3 + \sqrt{-1}x^4)$
3. $\mathrm{Cliff}(\mathbb{R}^4) = \bigoplus_{i=1}^{4} \mathbb{C}\Psi_i = \mathbb{C}^4$, $(\mathbb{C}^{2^{[4/2]}} = \mathbb{C}^4)$ where

$$\Psi_1 := \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad \Psi_2 := \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \quad \Psi_3 := \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad \Psi_4 := \begin{pmatrix}0\\1\\1\\0\end{pmatrix},$$

Clifford Module of $\mathbb{E}^4$

1. $\mathrm{Cliff}(\mathbb{R}^4)^* = \bigoplus_{i=1}^{4} \mathbb{C}\overline{\Psi}_i$ where $\overline{\Psi}_1 = (0,1,0,1)$, $\overline{\Psi}_2 = (1,0,1,0)$, $\overline{\Psi}_3 = (0,-1,1,0)$, and $\overline{\Psi}_4 = (1,0,0,-1)$,
2. $\mathrm{Cliff}(\mathbb{R}^4)$ can play a role of a "dual vector space" to $\mathbb{R}^4$, $(\mathrm{Cliff}^{pr}(\mathbb{R}^4) = \mathrm{Cliff}(\mathbb{R}^n))$:

$$\sum_i \overline{\Psi}_1 \gamma_{\{dx\}}(dx^i)\Psi_1 dx^i = 2dZ_1, \quad \sum_i \overline{\Psi}_2 \gamma_{\{dx\}}(dx^i)\Psi_2 dx^i = 2d\overline{Z}_1,$$

$$\sum_i \overline{\Psi}_3 \gamma_{\{dx\}}(dx^i)\Psi_3 dx^i = 2dZ_2, \quad \sum_i \overline{\Psi}_4 \gamma_{\{dx\}}(dx^i)\Psi_4 dx^i = 2d\overline{Z}_2.$$

$\Psi_1, \ldots, \Psi_4$ are regarded as the dual bases of $\mathbb{R}^4$.

Dirac operator $\not{D}_{\mathbb{E}^4}$ of $\mathbb{E}^4$

The solution of $\not{D}_{\mathbb{E}^4}\Psi = 0$ in $\Psi \in \mathcal{C}^\infty(\mathbb{E}^4, \mathbb{C}^4)$ over $\mathbb{E}^4$ is given by a constant function,

$$\Psi = \sum_{i=1}^{4} a_i \Psi_i, \quad a_i \in \mathbb{C}$$

where

$$\Psi_1 := \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Psi_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \Psi_3 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \Psi_4 := \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The solution space $\mathrm{Sol}(\not{D}_{\mathbb{E}^4}) = \mathrm{Ker}(\not{D}_{\mathbb{E}^4})$ is equal to $\mathbb{C}^4 = \mathrm{Cliff}(\mathbb{R}^4)$ and has the Clifford module structure.

A conformal surface S embedded in $\mathbb{E}^4$

A conformal surface S embedded in $\mathbb{E}^4$

1. The Riemannian metric of a conformal surface S

$$g_{S\alpha\beta} = \rho\delta_{\alpha\beta},$$

   where $\{ds^\alpha\}$ is an orthogonal system of S.
   The complex parameterization S : $dz := ds^1 \pm \sqrt{-1}ds^2$.
2. $Z := (Z^1, Z^2) : S \hookrightarrow \mathbb{E}^4 = \mathbb{C} \times \mathbb{C}$. $\quad Z(s) \in S \subset \mathbb{E}^4$.
3. The tubular neighborhood $(S_\varepsilon, g_{S\alpha\beta} \oplus \eta_{\perp ij})$ of S.
   The parameterization $\xi = (\xi^1, \xi^2, \xi^3, \xi^4) = (s^1, s^2, q^3, q^4)$.
4. $\not{D}_{\mathcal{M}\hookrightarrow\mathbb{E}^n} := \eta_{\|}^{1/4} \not{D}_{\mathbb{E}^n} \eta_{\|}^{-1/4}|_{\bigcap_{\mu=k+1}^n \mathrm{Ker}(\partial_{q^\mu}), q=0}$.

Dirac operator defined on a conformal surface $S \hookrightarrow \mathbb{E}^4$

Theorem 1: Submanifold Dirac operator $S \hookrightarrow \mathbb{E}^4$

The submanifold Dirac operator on a conformal surface $S \hookrightarrow \mathbb{E}^4$ is given by

$$\not{D}_{S\hookrightarrow\mathbb{E}^4} = 2\begin{pmatrix} & & \overline{p}_c & \partial \\ & & \overline{\partial} & -p_c \\ p_c & \partial & & \\ \overline{\partial} & -\overline{p}_c & & \end{pmatrix},$$

where $\partial := (\partial_{s^1} - \sqrt{-1}\partial_{s^2})/2$, $\overline{\partial} := (\partial_{s^1} + \sqrt{-1}\partial_{s^2})/2$ and p_c is

$$p_c := -\frac{1}{2}\rho^{1/2}\mathrm{tr}_{2\times 2}(\Gamma^{\alpha}_{3\beta} + \sqrt{-1}\Gamma^{\alpha}_{4\beta}).$$

where $\Gamma^{\alpha}_{a\beta}$ $(a = 3, 4)$ is the Weingarten map of S.

M. Pedit-Pinkall and Konopelchenko derived the same operator from geometrical investigation (1998,9).

A generalized Weierstrass relation for $S \hookrightarrow \mathbb{E}^4$

Consider the solution $\varphi \in \mathcal{C}^\infty(S, \mathbb{C}^4)$ of $\not{D}_{S\hookrightarrow\mathbb{E}^4}\varphi_a = 0$.

Theorem 2: Dirac operator defined on a conformal surface $S \hookrightarrow \mathbb{E}^4$

The solutions $\varphi_1 := \begin{pmatrix} f \\ g \\ 0 \\ 0 \end{pmatrix}$ $\varphi_2 := \begin{pmatrix} 0 \\ 0 \\ m \\ n \end{pmatrix} \in \mathcal{C}^\infty(S, \mathbb{C}^4)$ of the Dirac equation $\not{D}_{S\hookrightarrow\mathbb{E}^4}\varphi_a = 0$, satisfying $(|f|^2 + |g|^2)(|m|^2 + |n|^2) = \rho^{1/2}$, hold the relations,

$$dZ^1 = fmdz - gnd\overline{z}, \quad dZ^2 = f\overline{n}dz + g\overline{m}d\overline{z},$$

where Z_1 and Z_2 are the embedding $(Z^1, Z^2) : S \hookrightarrow \mathbb{C} \times \mathbb{C} \approx \mathbb{E}^4$.

This is the generalized Weierstrass relation.

Submanifold Dirac operator for $S \hookrightarrow \mathbb{E}^4$

Sketch of Proof:

1. For φ_a's, we have their partners, $\varphi_3 := \begin{pmatrix} -\overline{g} \\ \overline{f} \\ 0 \\ 0 \end{pmatrix}$ $\varphi_4 := \begin{pmatrix} 0 \\ 0 \\ -\overline{n} \\ \overline{m} \end{pmatrix}$.
2. From the construction of $\not{D}_{S\hookrightarrow\mathbb{E}^4}$ the solutions can be extended to the solutions φ of $\not{D}_{\mathbb{E}^4}\eta_{\|}^{-1/4}\varphi(x) = 0 \quad x \in S_\varepsilon$ because $\eta_\perp|_{q=0} = 1$ and $\eta_\perp^{-1/4}\partial_q\eta_\perp^{1/4}\Psi = 0$.
3. Since S_ε is diffeomorphic to $\mathbb{R}^4$, it must be a restriction of the constant functions Ψ from sheaf theory, and thus there is a section of principal bundle $\tau \in \mathcal{C}^\infty(S, \mathrm{CG}(\mathbb{R}^4))$ such that

$$\varphi_a(x) = \tau(x)\Psi_a(x) \quad x \in \mathcal{M}_\varepsilon.$$

5. By letting $\gamma_{S,\{d\xi\}}(d\zeta^\beta) = \sigma^\beta$, we embed it $\tau_{S,\mathbb{E}^2} : \mathrm{Cliff}(\mathbb{R}^2) \to \mathrm{Cliff}(\mathbb{R}^4)$ $(\tau_{S,\mathbb{E}^2}(\gamma_{S,\{d\xi\}}(d\zeta^\beta)) = \sigma^1 \otimes \sigma^\beta)$.
6. By introducing $\hat{\varphi}_1 := \varphi_1 + \varphi_2$, $\hat{\varphi}_2 := \varphi_3 + \varphi_4$, $\hat{\varphi}_3 := \varphi_1 + \varphi_4$, and $\hat{\varphi}_4 := \varphi_3 + \varphi_2$. we have $\hat{\varphi}_a = \rho^{1/2}\mathrm{e}^\Omega \Psi_a|_{q=0}$ $(a = 1, 2, 3, 4)$ by the spin matrix, $\rho^{1/2}\mathrm{e}^\Omega = \begin{pmatrix} f & -\overline{g} & 0 & 0 \\ g & \overline{f} & 0 & 0 \\ 0 & 0 & m & -\overline{n} \\ 0 & 0 & n & \overline{m} \end{pmatrix}$.
7. At $p \in \mathcal{M}$, we have

$$2dZ_1 = \overline{\hat{\varphi}}_1 \sigma_1 \otimes \sigma^\alpha \hat{\varphi}_1 ds^\alpha, \quad 2d\overline{Z}_1 = \overline{\hat{\varphi}}_2 \sigma_1 \otimes \sigma^\alpha \hat{\varphi}_2 ds^\alpha,$$

$$2dZ_2 = \overline{\tilde{\varphi}}_3 \sigma_1 \otimes \sigma^\alpha \tilde{\varphi}_3 ds^\alpha, \quad 2d\overline{Z}_2 = \overline{\tilde{\varphi}}_4 \sigma_1 \otimes \sigma^\alpha \tilde{\varphi}_4 ds^\alpha.$$

Explicit representation of them proves it. □

Submanifold Dirac Operator

Remark of Submanifold Dirac Operator

The submanifold Dirac operator has the data of embedding of the surface $\iota : S \hookrightarrow \mathbb{E}^4$ for every point $p \in S$.

For example,

$$Z^1 = \int^z dZ^1 = \int^z fmdz - gnd\bar{z},$$

$$Z^2 = \int^z dZ^2 = \int^z f\bar{n}dz + g\overline{m}d\bar{z}.$$

Submanifold Dirac Operator

Other Results of Study of Submanifold Dirac Operator

1. $n = 2, k = 1$: $C \hookrightarrow \mathbb{E}^2$: $\not{D}_{C\hookrightarrow\mathbb{E}^2} = \sqrt{-1}\begin{pmatrix} \partial_s & \frac{1}{2}k \\ \frac{1}{2}k & -\partial_s \end{pmatrix}$
   1. It is the same as the Dirac operator (one of Lax operator) of the **inverse scattering method** of **loop soliton** whose curvature obeys the **modified KdV equation.**
   2. If and only if the deformation of the curve C preserves the eigenvalue, $\partial_t E = 0$ (**adiabatic condition**) of the equation $\not{D}_{C\hookrightarrow\mathbb{E}^2}\psi = E\psi$, the time development of **the curve C is governed by the loop soliton whose curvature obeys the modified KdV equation** (M. Phys Rev. A 1993).
   3. If we put a Dirac particle into a thin curve, we realize **the system of the inverse scattering method of the modified KdV equation.**
   4. The analytic index of $\not{D}_{C\hookrightarrow\mathbb{E}^2}$ is the linking number of the curve.

Other Results of Study of Submanifold Dirac Operator

3. $n = 3, k = 1$: $C \hookrightarrow \mathbb{E}^3$: $\not{D}_{C\hookrightarrow\mathbb{E}^3} = \sqrt{-1}\begin{pmatrix} \partial_s & \frac{1}{2}\kappa \\ \frac{1}{2}\overline{\kappa} & -\partial_s \end{pmatrix}$,

   where $\kappa = k \exp\left(\sqrt{-1}\int^s ds\tau(s)\right)$ k : *curvature*, τ : *torsion*

   1. It is the same as the Dirac operator of the inverse scattering method of the **Hasimoto vortex soliton** whose complex curvature obeys the **nonlinear Schrödinger equation.**
   2. If and only if the deformation of the curve C preserves the eigenvalue, $\partial_t E = 0$ (**adiabatic condition**) of the equation $\not{D}_{C\hookrightarrow\mathbb{E}^3}\psi = E\psi$, the time development of **the curve C is governed by the Hasimoto soliton whose curvature obeys nonlinear Schrödinger equation** (M. J. Phys. A 1996).
   3. If we put a Dirac particle into a thin curve, we realize **the system of the inverse scattering method of nonlinear Schrödinger equation.**
   4. The index of $\not{D}_{C\hookrightarrow\mathbb{E}^3}$ is the linking number of the curve.

Submanifold Dirac Operator

Other Results of Study of Submanifold Dirac Operator

3. $n = 3, k = 2$ conformal surface $S \hookrightarrow \mathbb{E}^3$: $\not{D}_{S\hookrightarrow\mathbb{E}^3} = \begin{pmatrix} p & \partial \\ \bar{\partial} & -p \end{pmatrix}$,

   where $p := -\frac{1}{2}\rho^{1/2}\mathrm{tr}_{2\times 2}(\Gamma^{\alpha}_{3\beta})$. This appears in the constant mean curvature surface if p is constant.
   This is related to the modified Novikov-Veselov equation.
   Atiyha-Singer type index theorem of the $D_{S\hookrightarrow\mathbb{E}^3}$ provides topological invariance (Rev. Math. Phys. 1999).

Open Problems

The global and analytic properties of the Dirac operator basically is not studied well.
For a given embedding $\mathcal{M} \hookrightarrow \mathbb{E}^n$ we have $D_{\mathcal{M}\hookrightarrow\mathbb{E}^n}$ whereas for given $D_{\mathcal{M}\hookrightarrow\mathbb{E}^n}$, it is an open problem whether it comes from some embeding $\mathcal{M} \hookrightarrow \mathbb{E}^n$.

Thank you for your attention!