Statistical Mechanics of Elastic Curves: beyond Euler’s elastica
弾性曲線の統計力学：オイラーのエラスティカを超えて
第 24 回 沼津研究会—幾何，数理物理，そして量子論

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1. Self-Introduction
2. Elastica Problem
3. Statistical Mechanics of Elastica (Quantized Elastica)
   1. Infinitesimal isometric deformation
   2. Infinitesimal isoenergy deformation
   3. MKdV flow
   4. Hyperelliptic Curves
   5. Topological Properties
   6. Final Remarks
Self-Introduction

Shigeki Matsutani (NIT)

Statistical Mechanics of Elastic Curves: beyond Euler's elastica

March 7, 2017 3 / 81
Self-Introduction

Electric Devices:


Computational Fluid Dynamics:


Self-Introduction

Nano Materials


Self-Introduction

Mathemtical Physics : Submanifold Quantum Mechanics


Mathematical Physics: Submanifold Dirac operator


Statistical Mechanics of Elastica


4. S. Matsutani, Relations in a quantized elastica

5. S. Matsutani, Euler’s Elastica and Beyond, J. Geom. Symm. Phys 17 (2010) 45-86,

6. S. Matsutani and E. Previato, From Euler’s elastica to the mKdV hierarchy, through the Faber polynomials,
Self-Introduction

Algebraic Curve:


Self-Introduction
Self-Introduction

現代数学と技術との関わり合い

適用を通じて

代数学

近代数学

松谷茂樹

現代数学社
Elastica Problem

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Immersion of curve

Immersion of Curve

$$Z : S^1 \hookrightarrow \mathbb{C} \text{ smooth } (|\partial_s Z| = 1).$$

$s$ is arclength.

$$Z(s) = X(s) + iY(s),$$

$$t = \partial_s Z = e^{i\phi},$$

$$\phi \in C^\infty(\kappa^{-1} S^1, \mathbb{R}))$$

$$= \cos \phi + i \sin \phi$$

$$n = it = i\partial_s Z.$$
Elastica Problem

Immersion of curve

Curvature & Frenet-Serret relation

\[ \mathbf{t} := \partial_s Z, \quad \partial_s \mathbf{t} = k \mathbf{n}, \quad \partial_s \mathbf{n} = -k \mathbf{t}, \quad (\partial_s^2 Z = ik \partial_s Z), \quad (1) \]

\( k := \partial_s \phi : \text{curvature}; \ k = 1/\text{[curvature radius]} \).

Elastica Problem (James Bernoulli (1691))

To find the shape of elastica (ideal thin elastic rod) in a plane.
Elastica Problem

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Origin of Elastica

Leonardo da Vinci (1452-1519)
Leonardo da Vinci (1452-1519) drew the pictures of bent beams
Origin of Elastica

Galileo Galilei (1564-1654)
Origin of Elastica

Galileo Galilei (1564-1654) investigated bent beams:
It is a problem of cantilever.
James Bernoulli (1654-1705) proposed the Elastica problem: **To find the shape of elastica (ideal thin elastic rod) in a plane.**
James Bernoulli (1654-1705) found the fact that the elastic force is proportional to $k$ and the Lemniscate integral: $s = \int_X^1 \frac{dX}{\sqrt{1 - X^4}}$. 
James Bernoulli defined the Lemniscate curve of eight figure.

\[ (x^2 + y^2)^2 = 2a^2(x^2 - y^2) \]

\[ \phi_{\text{lemni}}: \text{tangential angle} \]

Elastica of Eight-Figure

\[ \phi_{\text{elas}}: \text{tangential angle} \]
James Bernoulli defined the Lemniscate curve of eight figure.

Lemniscate

\[(x^2 + y^2)^2 = 2a^2(x^2 - y^2)\]

\[\phi_{\text{lemni}}: \text{tangential angle}\]

\[\phi_{\text{lemni}} = \frac{3}{2}\phi_{\text{elas}} \quad [M \ 1995]\]
Elastica Problem

Daniel Bernoulli (1700-1782)
Daniel Bernoulli (1700-1782) discovered the least principle 1738 in a letter to Euler (1707-1783).

An elastica is realized as the least point of the energy, i.e.,

**Euler-Bernoulli energy**

\[
\mathcal{E}[Z] := \int_{S^1} k^2(s) \, ds = \int_{S^1} (\partial_s \phi(s))^2 \, ds \\
= \int_{S^1} \{Z, s\}_{SD} \, ds \\
= \int_{S^1} g^{-1} dg * g^{-1} dg, \quad g \in U(1)
\]

\[
\{Z, s\}_{SD}:\text{Schwarz derivative}
\]

**Elastica problem is the oldest harmonic map problem.**
Leonhard Euler (1707-1783)
Euler’s solution

By discovering the variational principle, he published the book “Method” 1744. In its Appendix, he completely solved the problems in terms of

1. **Elliptic integrals**,  
2. Moduli of elliptic curves,  
Elastica Problem

Euler’s solution

\[ s = \int X \frac{\lambda^2 dX}{\sqrt{\lambda^4 - (\alpha + \beta X + \gamma X^2)^2}}, \]

\[ Y = \int X \frac{(\alpha + \beta X + \gamma X^2) dX}{\sqrt{\lambda^4 - (\alpha + \beta X + \gamma X^2)^2}}. \]

\[ ak + \frac{1}{2} k^3 + \partial_s^2 k = 0. \]

Euler relation (M-Previato 2014)

\[ X(s) - X_0 = \frac{1}{4} k(s) \]

affine coordinate \( \propto \) affine connection
Elastica Problem

Euler’s list of Elastica

\[ \text{Diagram of Euler’s list of Elastica} \]
Statistical Mechanics of Elastica

Statistical Mechanic's of Elastic Curves: beyond Euler's elastica
Pictures of DNAs by atomic force microscopes shows the super-coils.

Pictures of DNAs by atomic force microscopes
These shapes are **super-coils** rather than **double coils**. Super-coil is weakly governed by the elastic force! But it is **not** realized as the **least point of the Euler-Bernoulli energy**, it is out of Euler’s list maybe due to the **thermal effect**!
Statistical Mechanics of Elastica

Statistical Mechanics of Elastica is to evaluate the state with the Boltzmann weight $e^{-\mathcal{E}[Z] \beta} (\beta > 0)$, i.e., the partition function,

$$Z[\beta] = \int_{\mathcal{M}} DZ \exp(-\beta \mathcal{E}[Z])$$

Here $\mathcal{M}$ is the set of the loops in the plane,

$$\mathcal{M} := \{ Z : S^1 \hookrightarrow \mathbb{C} \mid Z \in C^\omega(S^1, \mathbb{C}), |dZ/ds| = 1 \},$$

$$\text{pr}_1 : \mathcal{M} \rightarrow \mathcal{M} := \mathcal{M}/\sim,$$

where $\sim$ means the equivalence coming from the euclidean move.
Purpose of Statistical Mechanics of Elastica

To find the natural topology and measure of $\mathcal{M}$ using the Boltzmann weight $e^{-\varepsilon[Z]_\beta}$.
As its first step, we consider the geometrical structure of $\mathcal{M}$.

Approach in Statistical Mechanics of Elastica

To find the geometrical structure of $\mathcal{M}$,
1) we consider the geometrical structure of its tangent space $T_Z\mathcal{M}$ at $Z \in \mathcal{M}$ as an infinitesimal deformation
2) using the data of $T_Z\mathcal{M}$ and its orbit, we classify $\mathcal{M}$ itself.
(M-Ônishi 2003, M-Previato 2016)

Notations

$\mathcal{A}^p(K) : K$-valued analytic $p$-form over $S^1$
($K$ is $\mathbb{R}$ or $\mathbb{C}$.)
Menu: Statistical Mechanics of Elastica (Quantized Elastica)

1. Infinitesimal isometric deformation
2. Infinitesimal isoenergy deformation
3. MKdV flow
4. Hyperelliptic Curves
5. Topological Properties
6. Final Remarks
Statistical Mechanics of Elastica

Infinitesimal Isometric Deformation
Tangent space $T_Z\mathcal{M}$ (= infinitesimal deformation)

To observe $T_Z\mathcal{M}$ at $Z \in \mathcal{M}$, we consider the deformation of the deformation parameter $t \in [0, \varepsilon)$ ($\varepsilon > 0$),

$$\partial_t Z(s) = v(s)\partial_s Z(s), \quad s \in S^1, \quad v \in \mathcal{A}^0(\mathbb{C}),$$

$$(v = v^{(r)} + iv^{(i)}, \quad v^{(r)}, v^{(i)} \in \mathcal{A}^0(\mathbb{R}))$$
Infinitiesmal Isometric Deformation

Tangent space $T_Z \mathcal{M}$

**Proposition**

At $Z \in \mathcal{M}$, the isometric deformation $([\partial_s, \partial_t]Z = 0)$ is reduced to two equations (Goldstein-Petrichi)

\[ \partial_t k = \Omega^{(II)} v^{(i)}, \quad (2) \]
\[ k v^{(i)} = \partial_s v^{(r)}. \quad (3) \]

where

\[ \Omega^{(II)} := \partial_s^2 + \partial_s (k \partial_s^{-1} k), \]
Proof of Proposition

\[[\partial_s, \partial_t]Z = 0\] means
proof of Proposition

\[ [\partial_s, \partial_t]Z = 0 \text{ means} \]

\[
\begin{align*}
\partial_s \partial_t Z & = \partial_s (\nu \partial_s Z) \\
& = (\partial_s \nu + i k \nu) \partial_s Z \\
\partial_t \partial_s Z & = \partial_t (e^{i \phi(s,t)}) \\
& = i (\partial_t \phi) \partial_s Z
\end{align*}
\]
proof of Proposition

\[ [\partial_s, \partial_t]Z = 0 \] means

\[
\begin{align*}
\partial_s \partial_t Z &= \partial_s (\nu \partial_s Z) \\
&= (\partial_s \nu + ik\nu) \partial_s Z \\
\partial_t \partial_s Z &= \partial_t (e^{i\varphi(s,t)}) \\
&= i(\partial_t \varphi) \partial_s Z
\end{align*}
\]

Thus \( i\partial_t \varphi = (\partial_s \nu + ik\nu) = (\partial_s \nu^{(r)} - k\nu^{(i)}) + i(\partial_s \nu^{(i)} + k\nu^{(r)}) \)
Infinitesimal Isometric Deformation

proof of Proposition

\[ [\partial_s, \partial_t]Z = 0 \text{ means} \]

\[
\begin{align*}
\partial_s \partial_t Z &= \partial_s (\nu \partial_s Z) \\
&= (\partial_s \nu + i k \nu) \partial_s Z \\
\partial_t \partial_s Z &= \partial_t (e^{i \phi(s,t)}) \\
&= i (\partial_t \phi) \partial_s Z
\end{align*}
\]

Thus \( i \partial_t \phi = (\partial_s \nu + i k \nu) = (\partial_s \nu^{(r)} - k \nu^{(i)}) + i (\partial_s \nu^{(i)} + k \nu^{(r)}) \)

Real part: \( \partial_s \nu^{(r)} - k \nu^{(i)} = 0 \rightarrow \nu^{(r)} = \partial_s^{-1} k \nu^{(i)} \)
Infinitesimal Isometric Deformation

proof of Proposition

\[ \partial_s \partial_t Z = \partial_s (\nu \partial_s Z) = (\partial_s \nu + ik \nu) \partial_s Z \]
\[ \partial_t \partial_s Z = \partial_t (e^{i\varphi(s,t)}) = i(\partial_t \varphi) \partial_s Z \]

Thus \( i\partial_t \varphi = (\partial_s \nu + ik \nu) = (\partial_s \nu^{(r)} - k \nu^{(i)}) + i(\partial_s \nu^{(i)} + k \nu^{(r)}) \)

Real part: \( \partial_s \nu^{(r)} - k \nu^{(i)} = 0 \rightarrow \nu^{(r)} = \partial_s^{-1} k \nu^{(i)} \)

Imaginary part: \( \partial_t k = \partial_s \partial_t \varphi = \partial_s (\partial_s \nu^{(i)} + k \nu^{(r)}) \)
proof of Proposition

\[ [\partial_s, \partial_t] Z = 0 \text{ means} \]

\[
\begin{align*}
\partial_s \partial_t Z &= \partial_s (\nu \partial_s Z) \\
&= (\partial_s \nu + i k \nu) \partial_s Z \\
\partial_t \partial_s Z &= \partial_t (e^{i \varphi(s,t)}) \\
&= i (\partial_t \varphi) \partial_s Z \\
\end{align*}
\]

Thus \(i \partial_t \varphi = (\partial_s \nu + i k \nu) = (\partial_s \nu^{(r)} - k \nu^{(i)}) + i(\partial_s \nu^{(i)} + k \nu^{(r)})\)

Real part: \(\partial_s \nu^{(r)} - k \nu^{(i)} = 0 \rightarrow \nu^{(r)} = \partial_s^{-1} k \nu^{(i)}\)

Imaginary part: \(\partial_t k = \partial_s \partial_t \varphi = \partial_s (\partial_s \nu^{(i)} + k \nu^{(r)})\)
\[
\begin{align*}
\partial_t k &= \partial_s (\partial_s + k \partial_s^{-1} k) \nu^{(i)} \\
\end{align*}
\]
proof of Proposition

\[ [\partial_s, \partial_t] Z = 0 \text{ means} \]

\[
\begin{align*}
\partial_s \partial_t Z &= \partial_s (\nu \partial_s Z) \\
&= (\partial_s \nu + i k \nu) \partial_s Z \\
\partial_t \partial_s Z &= \partial_t (e^{i \phi(s,t)}) \\
&= i (\partial_t \phi) \partial_s Z
\end{align*}
\]

Thus \( i \partial_t \phi = (\partial_s \nu + i k \nu) = (\partial_s \nu^{(r)} - k \nu^{(i)}) + i(\partial_s \nu^{(i)} + k \nu^{(r)}) \)

Real part: \( \partial_s \nu^{(r)} - k \nu^{(i)} = 0 \rightarrow \nu^{(r)} = \partial_s^{-1} k \nu^{(i)} \)

imaginary part: \( \partial_t k = \partial_s \partial_t \phi = \partial_s (\partial_s \nu^{(i)} + k \nu^{(r)}) \)

\( \partial_t k = \partial_s (\partial_s + k \partial_s^{-1} k) \nu^{(i)} \)
**Proposition**

At $Z \in \mathbb{M}$, the isometric deformation $([\partial_s, \partial_t]Z = 0)$ is reduced to two equations (Goldstein-Petrichi)

\[
\begin{align*}
\partial_t k &= \Omega^{(ll)} v^{(i)}, \\
k v^{(i)} &= \partial_s v^{(r)}.
\end{align*}
\]  

where

\[
\Omega^{(ll)} := \partial_s^2 + \partial_s (k \partial_s^{-1} k),
\]
Infinitesimal Isometric Deformation

Eq. (3) \( \partial_s v(r) = k v(i) \)

1. In order to find the space satisfying Eq. (3), we consider the map \( \ell_d \),

\[ \ell_d : \mathcal{A}^0(\mathbb{R}) \to \mathcal{A}^1(\mathbb{R}), \quad \ell_d(v(i)) = k v(i) ds, \]

2. Let the inverse image of \( d\mathcal{A}^0(\mathbb{R}) \subset \mathcal{A}^1(\mathbb{R}) \) by \( \ell_d \) be \( \hat{\mathcal{A}}^0(\mathbb{R}) := \ell_d^{-1} d\mathcal{A}^0(\mathbb{R}) \), which is the space satisfying Eq. (3).

Eq. (3) \( \partial_s v(r) = k v(i) \), \( v = v(r) + i v(i) \)

\[ \ell_r^0 : \hat{\mathcal{A}}^0(\mathbb{R}) \to \mathcal{A}^0(\mathbb{R}) : (v(i) \mapsto v(r)) \text{ because of } \partial_s v(r) = k v(i), \]

\[ v(r) = \ell_r^0(v(i)) = \int_0^s k v(i) ds = \int_0^s \partial_s v(r) ds \]

\[ \ell : \hat{\mathcal{A}}^0(\mathbb{R}) \to \mathcal{A}^0(\mathbb{C}) ; (v(i) \mapsto v = v(r) + i v(i) = \ell_r^0(v(i)) + i v(i)). \]
Tangent space $T_Z \mathcal{M}$ (Space of infinitesimal isometric deformation)

For a point $Z \in \mathcal{M}$, we have

\[ \ell : \hat{A}^0(\mathbb{R}) \to A^0(\mathbb{C}) ; (\ell(v^{(i)})) = v = v^{(r)} + iv^{(i)} \]

\[ \text{pr}_1 : \mathcal{M} \to \mathbb{M} := \mathcal{M}/\sim, \]

$\sim$ means the equivalence coming from the euclidean move.
Tangent space $T_Z M$ (Space of infinitesimal isometric deformation)

For a point $Z \in \mathcal{M}$, we have

$$\ell : \hat{A}^0(\mathbb{R}) \rightarrow A^0(\mathbb{C}) ; (\ell(v^{(i)})) = v = v^{(r)} + iv^{(i)}$$

**Proposition**

(Brylinski) For a point $Z \in \mathcal{M}$, we have the map $\ell$ induces the bijection $\ell^#$ and the surjection $\ell^b :$

$$\mathcal{A}^0(\mathbb{R})/\mathbb{R} \cong \hat{A}^0(\mathbb{R}) \xrightarrow{\ell^#} T_Z(\mathcal{M})$$

$$\xymatrix{T_{pr_1}(Z)(\mathcal{M}) \ar@{^{(}->}[rr]^{pr_{1*}} \ar[rru]^\ell \ar[rrd]_{\ell^b} & & T_Z(\mathcal{M})}$$
Infinitesimal Isometric Deformation

Tangent space $T_ZM$ (Space of infinitesimal isometric deformation)

1. **Translation of SE(2)**
   
   Since $\partial_t Z = c = c_1 + ic_2 \in \mathbb{C}$ means the translation,
   
   if $v = \frac{c}{\partial_s Z} = c_1 \cos \phi + c_2 \sin \phi - ic_1 \sin \phi + ic_2 \cos \phi$,
   
   it vanishes at $M$.

   In fact $v^{(i)} = -c_1 \sin \phi + c_2 \cos \phi$ corresponds to
   
   $v^{(r)} = \int_0^s k v^{(i)} \, ds = c_1 \cos \phi + c_2 \sin \phi$,
   
   $\partial_t Z = \left( v + \frac{c}{\partial_s Z} \right) \partial_s Z$ means translation

2. **Rotation of SE(2)**
   
   $\partial_t Z = c' \partial_s Z$, $c' \in \mathbb{R}$ means the rotation.

   It implies $Z = Z(s + c't)$ or $\partial_s Z = e^{i\phi(s+c't)} = e^{i\phi(s)+i\phi_0}$

   It corresponds to $v^{(r)} \rightarrow v^{(r)'} = v^{(r)} + c'$ of $\mathcal{L}_r^0$. 
Infinitesimal Isometric Deformation

Tangent space $T_Z \mathcal{M}$ (Space of infinitesimal isometric deformation)

**Proposition**

For a point $Z \in \mathcal{M}$, the map $\ell^b$ induces the bijection $\ell^b :$

$$\mathcal{A}^0(\mathbb{R})/(\mathbb{R} \oplus (\mathbb{R} \cos \phi + \mathbb{R} \sin \phi)) \cong T_{\text{pr}1}(Z)(\mathcal{M})$$
Infinitesimal Isoenergy Deformation
To consider the effect of energy $E(>0)$, we introduce

$$\mathcal{M}_E := \{Z \in \mathcal{M} \mid \mathcal{E}[Z] = E\}.$$ 

To investigate this geometric structure, we consider the subset of $T_Z \mathcal{M}$ which preserves the energy, i.e., infinitesimal isoenergy deformation:

$$\text{pr}_{1,E} : \mathcal{M}_E \to \mathbb{M}_E := \mathcal{M}_E / \sim$$
Proposition

At $Z \in \mathcal{M}$, the deformation is isoenergy, i.e., $\partial_t E(Z) = 0$, if and only if $\partial_t k \in \hat{A}^0(\mathbb{R})$.

proof

$$\partial_t E(Z) = \partial_t \int k^2 ds = 2 \int k \partial_t k ds = \int \partial_s \hat{f} ds = 0$$

because from the condition $\partial_t k \in \hat{A}^0(\mathbb{R})$, $k \partial_t k ds \in dA^0(\mathbb{R})$, i.e.,

$$k \partial_t k = \partial_s \hat{f} / 2, \ (f \in A^0(\mathbb{R}))$$
\[ \partial_t Z \text{ isometric deformation in } t \]

\[ \iff \]

i) \( v^{(i)} \in \hat{A}^0(\mathbb{R}) \)

ii) \( \partial_t k = \Omega^{(II)} v^{(i)} \)
Infinitesimal Isoenergy Deformation

1. \( \partial_t Z \) isometric deformation in \( t \)
   \[ \iff \]
   i) \( v^{(i)} \in \hat{A}^0(\mathbb{R}) \)
   ii) \( \partial_t k = \Omega^{(II)} v^{(i)} \)

2. \( \partial_t Z \) isometric and isoenergy deformation in \( t \)
   \[ \iff \]
   i) \( v^{(i)} \in \hat{A}^0(\mathbb{R}) \)
   ii) \( \partial_t k = \Omega^{(II)} v^{(i)} \)
   iii) \( \partial_t k \in \hat{A}^0(\mathbb{R}) \)
Infinitesimal Isoenergy Deformation

1. \( \partial_t Z \) isometric deformation in \( t \)
   \[ \Leftrightarrow \quad \begin{align*}
   & \text{i) } \nu(i) \in \hat{A}^0(\mathbb{R}) \\
   & \text{ii) } \partial_t k = \Omega(\II) \nu(i)
   \end{align*} \]

2. \( \partial_t Z \) isometric and isoenergy deformation in \( t \)
   \[ \Leftrightarrow \quad \begin{align*}
   & \text{i) } \nu(i) \in \hat{A}^0(\mathbb{R}) \\
   & \text{ii) } \partial_t k = \Omega(\II) \nu(i) \\
   & \text{iii) } \partial_t k \in \hat{A}^0(\mathbb{R})
   \end{align*} \]

3. \( \Rightarrow \) there might be another isometric deformation in another time \( t' \)
   \[ \Leftrightarrow \quad \begin{align*}
   & \text{i) } \partial_t k \in \hat{A}^0(\mathbb{R}) \\
   & \text{ii) } \partial_{t'} k = \Omega(\II) \partial_t k
   \end{align*} \]
Infinitesimal Isoenergy Deformation

1. $\partial_t Z$ isometric deformation in $t$
   \[ \Leftrightarrow \]
   i) $\nu^{(i)} \in \hat{A}^0(\mathbb{R})$
   ii) $\partial_t k = \Omega^{(II)}\nu^{(i)}$

2. $\partial_t Z$ isometric and isoenergy deformation in $t$
   \[ \Leftrightarrow \]
   i) $\nu^{(i)} \in \hat{A}^0(\mathbb{R})$
   ii) $\partial_t k = \Omega^{(II)}\nu^{(i)}$
   iii) $\partial_t k \in \hat{A}^0(\mathbb{R})$

3. $\Rightarrow$ there might be another isometric deformation in another time $t'$
   \[ \Leftrightarrow \]
   i) $\partial_t k \in \hat{A}^0(\mathbb{R})$
   ii) $\partial_{t'} k = \Omega^{(II)}\partial_t k = \Omega^{(II)^2}\nu^{(i)}$
Infinitesimal Isoenergy Deformation

1. $\partial_t Z$ isometric deformation in $t$
   \[\Leftrightarrow\]
   i) $v(i) \in \hat{A}^0(\mathbb{R})$
   ii) $\partial_t k = \Omega^{(II)} v(i)$

2. $\partial_t Z$ isometric and isoenergy deformation in $t$
   \[\Leftrightarrow\]
   i) $v(i) \in \hat{A}^0(\mathbb{R})$
   ii) $\partial_t k = \Omega^{(II)} v(i)$
   iii) $\partial_t k \in \hat{A}^0(\mathbb{R})$

3. $\Rightarrow$ there might be another isometric deformation in another time $t'$
   \[\Leftrightarrow\]
   i) $\partial_t k \in \hat{A}^0(\mathbb{R})$
   ii) $\partial_{t'} k = \Omega^{(II)} \partial_t k = \Omega^{(II)^2} v(i)$

   $\Rightarrow$ These induce a certain hierarchy.
Proposition

If for $v^{(i)} \in A^0(\mathbb{R})$, $\{\Omega^{(II)^n} v^{(i)}\}_{n=0,1,2,...}$ belong to $\hat{A}^0(\mathbb{R})$ the parameters $(\tilde{t}_1, \tilde{t}_2, \ldots) \in [0, \varepsilon)$, preserves the induced metric and the energy, and we have a sequence

$$\partial_{\tilde{t}_1} k = \Omega^{(II)} v^{(i)},$$

$$\partial_{\tilde{t}_2} k = \Omega^{(III)} \partial_{\tilde{t}_1} k = \Omega^{(II)^2} v^{(i)},$$

$$\partial_{\tilde{t}_3} k = \Omega^{(III)} \partial_{\tilde{t}_2} k = \Omega^{(II)^2} \partial_{\tilde{t}_1} k = \Omega^{(III)^2} v^{(i)},$$

$$\vdots$$
MKdV flow
Infinitesimal Isoenergy Deformation

Tangent space $T_Z\mathcal{M}$

**Lemma**

*For $c \in \mathbb{R}$ and $Z \in \mathcal{M}$, the static (trivial) deformation, $Z(s + ct)$, is generated by*

$$\partial_t Z = c \partial_s Z, \iff \partial_t k = c \partial_s k.$$ 

**Proposition**

*For the static deformation, $\mathcal{M}/U(1)$ is stable, and the static deformation in $\mathcal{M}$ is isometric and isoenergy.*
Proposition

For $Z \in \mathbb{M}$ and $k := k[Z]$, we consider static deformation,

$$\partial_{t_1} k = \partial_s k,$$

and then we have the following isometric and isoenergy relations:

$$\partial_{t_2} k = \Omega^{(II)} \partial_{t_1} k = \Omega^{(II)} \partial_s k,$$

$$\partial_{t_3} k = \Omega^{(II)} \partial_{t_2} k = \Omega^{(II)^2} \partial_{t_1} k = \Omega^{(II)^2} \partial_s k,$$

$$\partial_{t_4} k = \Omega^{(II)} \partial_{t_2} k = \Omega^{(II)^2} \partial_{t_2} k = \Omega^{(II)^3} \partial_s k,$$

$$\vdots$$

These agree with the MKdV hierarchy.
Since the MKdV hierarchy is integrable, we can consider the orbits in $\mathcal{M}$, $\mathcal{M}$, $\mathcal{M}_E$ and $\mathcal{M}_E$:

1. These orbits induce their **orbital decomposition**.
2. These orbits are described by **hyperelliptic functions** and **moduli space of hyperelliptic curves**.

→ We partially find their geometrical structure.
Open problems

1) The solution space contains Euler’s results as genus one.
2) The solution of MKdV hierarchy is given by the hyper-elliptic curves including $\infty$ genus.
Abelian Function Theory
"The elliptic function theory is to study the algebraic properties of elliptic curves, the analytic properties of their abelian functions (=elliptic functions), and these relations."

\[ y^2 = (x - b_0)(x - b_1)(x - b_2) \]

standard form

Algebraic Properties

\[ \sigma \text{-func} / \text{entire func over } \mathbb{C} \]

Analytic Properties

Aim of the study of Abelian Functions

As the elliptic function theory has a power to various fields of mathematics, physics, engineer as concrete theory of functions, we want to construct the Abelian function theory which has concrete and abstract expressions in order that it has a power to various fields.
Weierstrass Normal Form

$(X, P)$: Pointed Riemann surface $P = \infty$

$(X, P)$ is characterized by the Wierstrass gap sequence, which is given by the numerical semi-group.

$(X, \infty)$: $(r, s) = 1$,

$$y^r + A_1(x)y^{r-1} + \cdots + A_{r-1}(x)y + A_r(x) = 0$$

where $A_j(x)$ ($j = 1, \ldots, r - 1$) whose order is $j < js/r$. $A_r(x)$ is a $s$-order polynomial.
Jacobi inversion formulae
Abelian function theory for Hyperelliptic curves

As the Euler’s elastica is related to elliptic function, the quantized elastica is related to the hyperelliptic function, (2003 MO, 2001, 2002 M), and naturally contains the Euler’s elastica.

A hyperelliptic curve $C_g$ of genus $g$ ($g > 0$) is given by,

$$y^2 = (x - b_1)(x - b_2) \cdots (x - b_{2g+1}),$$

where $b_j$’s are complex numbers.

$g = 1$ case

$g = 2$ case
Abelian function theory for Hyperelliptic curves

Hyperelliptic Integrals

Hyperelliptic complete integrals:

\[ \omega'_{ij} := \int_{\alpha_i} \nu^l_j, \quad \omega''_{ij} := \int_{\beta_i} \nu^l_j, \quad i, j = 1, \ldots, g, \]

\[ \eta'_{ij} := \int_{\alpha_i} \nu^{ll}_j, \quad \eta''_{ij} := \int_{\beta_i} \nu^{ll}_j, \quad i, j = 1, \ldots, g, \]

where hyperelliptic differentials, 1st and 2nd kinds:

\[ \nu^l_i = \frac{x^{i-1}dx}{2y}, \quad \nu^{ll}_i = \frac{(x^{g+i-1} + \sum_{j=1}^{g+i-2} a_{ij}x^j)dx}{2y}. \]

for certain \( a_{ij} \) of \( b_i \)'s, \( i = 1, \ldots, g \).
### Symplectic structure as Legendre relations

Legendre relations as the symplectic structure:

\[ \omega' \eta'' - \omega'' \eta' = \frac{\pi}{2} \sqrt{-1} l_g \]

This is the same as a part of Galois’s letter to A. Chevalier:
Abelian function theory for Hyperelliptic curves

**Hyperelliptic Jacobian**

For a symmetric product space of $C_g$, $S^g(C_g)$, the Abelian map is defined by

$$u := (u_1, \cdots, u_g) : S^g(C_g) \rightarrow \mathbb{C}^g,$$

$$u_k((x_1, y_1), \cdots, (x_g, y_g)) := \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} \frac{x^{k-1}dx}{2y}.$$  

The hyperelliptic Jacobian:

$$\mathcal{J}_g = \mathbb{C}^g / \Lambda, \quad \Lambda = \langle \omega', \omega'' \rangle_{\mathbb{Z}}.$$
Abelian function theory for Hyperelliptic curves

theta function and sigma function

\[ T = \omega' \omega'' \] is given by

\[ \theta \begin{bmatrix} a \\ b \end{bmatrix} (z) = \theta \begin{bmatrix} a \\ b \end{bmatrix} (z; T) \]

\[ = \sum_{n \in \mathbb{Z}^g} \exp \left[ 2\pi i \left\{ \frac{1}{2} t(n + a) T(n + a) + t(n + a)(z + b) \right\} \right] \]

for \( g \)-dimensional complex vectors \( a \) and \( b \).

The \( \sigma \)-function is given by

\[ \sigma(u) = \gamma_0 \exp \left\{ -\frac{1}{2} t u \eta' \omega' \omega'^{-1} u \right\} \vartheta \left[ \delta'' \right] \left( \frac{1}{2} \omega' \omega'^{-1} u; T \right) \]

where \( \delta \) and \( \delta' \) are half-integer characteristics.
Abelian function theory for Hyperelliptic curves

\( \phi \) function

\[
\phi_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u),
\]

\[
\zeta_i = \frac{\partial}{\partial u_i} \log \sigma(u)
\]

\[
al_r := \sqrt{(b_r - x_1)(b_r - x_2) \cdots (b_r - x_g)} = \gamma_0 e^{-\eta_r u} \frac{\sigma(u + \omega_r)}{\sigma(\omega_r) \sigma(u)},
\]
Euler’s results from a modern point of view

Euler's elastica and symplectic structure

\[ Z(s) = (-\zeta(s) + (a/6)s)/i, \]

The symplectic structure in Jacobian is given by

\[ \langle ds, \zeta(s)ds \rangle = 1 \]

and

\[ \omega' \eta'' - \omega'' \eta' = \frac{\pi}{2}i. \]

It means that for the space

\[ G := \{(s, Z(s)) | s \in S^1\} \subset S^1 \times Z(S^1) \]

\( T^*G \) has the “symplectic structure” \( ds \wedge dZ \).
Theorem (2002, 2010 M) 1) For the hyperelliptic curve $C_g$, by letting $s := u_g$, $Z_r \in \mathcal{M}_{\text{elas}, E}^\mathbb{C}$ ($r = 1, 2, \cdots, 2g + 1$) is given by

$$\partial_s Z_r(s) = a_1 r(s)^2, \quad Z_r(s) = b_r^g s - \sum_{i=1}^{g} \zeta_i(s) b_r^{i-1}.$$  

2) $Z_r(u \in \mathcal{J}_g)$ is isoenergy flows!!!

3) The energy is given by the hyperelliptic integrals:

$$\int_{\alpha_a} k_r^2 ds = -4\eta_{ag}' + 2(\lambda_{2g} + b_r)\omega_{ag}'$$

4) $\text{Vol}(\mathcal{M}_{\text{elas}, E}^\mathbb{C})$ is the volume of the real subspace in the Jacobi variety $\mathcal{J}_g$. 

Hyperelliptic Solutions and Quantized Elastica
Hyperelliptic Solutions and Quantized Elastica

Remark 1) The shape of quantized elastica is

\[ Z_r(s) = b^g_r s - \sum_{i=1}^{g} \zeta_i(s) b^{i-1}_r, \]

whereas that of Euler’s elastica is

\[ Z'(s) = (a/6)s - \zeta(s) \text{ for } (Z'(s) = Z(s)/\sqrt{-1}). \]

2) The energy of quantized elastica is

\[ \int k^2 ds = -4\eta'_{ag} + 2(\lambda_{2g} + b_r)\omega'_{ag}, \]

whereas that of Euler’s elastica is

\[ \int k^2 ds = -4\eta' + 2(e_1)\omega'. \]

3) The generalization of Euler’s relation is

\[ Z(u) - Z(u - \omega) = \sum_i^g b^{i-1}_i \partial_i \log \partial_{t_1} Z. \]
Hyperelliptic Solutions and Quantized Elastica

**Remark**

4) The shape of quantized elastica is

\[
\begin{pmatrix}
  Z_1 \\
  Z_2 \\
  \vdots \\
  Z_{g+1}
\end{pmatrix} = \begin{pmatrix}
  b_1^g & b_1^{g-1} & b_1^{g-2} & \cdots & b_1 & 1 \\
  b_2^g & b_2^{g-1} & b_2^{g-2} & \cdots & b_2 & 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  b_{g+1}^g & b_{g+1}^{g-1} & b_{g+1}^{g-2} & \cdots & b_{g+1} & 1
\end{pmatrix}
\begin{pmatrix}
  s \\
  \zeta_g \\
  \vdots \\
  \zeta_1
\end{pmatrix}.
\]

\[
\langle \zeta_r \, dt_{g-r}, \, dt_v \rangle = \delta_{r,v} \text{ means } \sum_{i} \pi_{r,i} Z_i dt_{g-r}, \, dt_v = \delta_{r,v},
\]

which is a "symplectic structure" in \( \mathcal{M}^\mathbb{C}_{\text{elas}} \).
Topological Properties
Lemma (Maclachlan)

The modulus space of conformal equivalence classes of compact Riemann surfaces of genus $g$ is simply connected.

MKdV hierarchy

For $\mathcal{M}^{\mathbb{C}}_{\text{elas},g} \rightarrow \mathcal{M}_{\text{elas},g}$, $(Z(\mathbb{T}^g) \mapsto \text{pt})$, we have

$$\mathcal{M}_{\text{elas},g} \subset \mathcal{M}_{\text{hyp},g}, \quad \mathcal{M}_{\text{hyp},g} \sim \text{pt.}$$
Lemma (MO 2003)

Due to the relations $\mathcal{M}_{\text{elas},g} \setminus \mathcal{M}_{\text{elas},g-1} \sim \mathbb{T}^{g-1}$ and

$$pt \leftrightarrow S^1 \leftrightarrow \mathbb{T}^2 \leftrightarrow \mathbb{T}^3 \leftrightarrow \mathbb{T}^4 \leftrightarrow \mathbb{T}^5 \leftrightarrow \ldots,$$

we have

$$\mathcal{M}_{\text{elas},1} \leftrightarrow \mathcal{M}_{\text{elas},2} \leftrightarrow \mathcal{M}_{\text{elas},3} \leftrightarrow \ldots.$$
Theorem (Bott-Tu)

The cohomology of the loop space $\Omega S^n$ over $S^n$ is given by

$$H^p(\Omega S^n, \mathbb{R}) = \mathbb{R}\delta_p \mod (n-1), 0.$$ 

For $n = 2$ case, the ring structure is given by

$$H^*(\Omega S^2, \mathbb{R}) = \mathbb{R}[x]/(x^2) \cdot \mathbb{R}[e],$$

where degree($e$) = 2 and degree($x$) = 1.

$$H^*(\Omega S^2, \mathbb{R}) = \mathbb{R} + \mathbb{R}x + \mathbb{R}e + \mathbb{R}xe + \mathbb{R}e^2 + \mathbb{R}xe^2 + \cdots.$$
A loop space

Since $\mathcal{M}_{\text{elas}}^C$ is topologically decomposed by genus, we have:

**Theorem (MO 2003)**

For the forgetful functor $\text{for} : \text{Diff} \to \text{Top}$, we have

$$H^*(\Omega S^2, \mathbb{R}) = H^*(\text{for}(\mathcal{M}_{\text{elas}}^C), \mathbb{R})$$

i.e., for $H^*(\Omega S^2, \mathbb{R}) = \mathbb{R}[x]/(x^2) \cdot \mathbb{R}[e]$, $H^*(\text{for}(\mathcal{M}_{\text{elas}}^C), \mathbb{R}) = \Lambda_{\mathbb{R}}[dt_1, e]$, where $\Lambda_{\mathbb{R}}[dt_1, e]$ is a ring generated by $dt_1$ and

$$e = dt_1 + dt_2 \wedge (dt_1 i_{\partial_1}) + dt_3 \wedge (dt_1 i_{\partial_1}) + \cdots$$

with the wedge product and the degree: $\text{degree}(dt_i) = 1$:

$$H^*(\text{for}(\mathcal{M}_{\text{elas}}^C), \mathbb{R}) = \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}e + \mathbb{R}e dt_1 + \mathbb{R}e^2 + \mathbb{R}e^2 dt_1 + \cdots$$
Proof:

Since $\epsilon \cdot 1 = dt_1$, and $\epsilon^{n-1} \cdot dt_1 = \epsilon^n \cdot 1 = dt_n \wedge dt_{n-1} \wedge \cdots \wedge dt_2 \wedge dt_1$, we have

$$\Lambda_R[dt_1, \epsilon] = R + R dt_1 + R \epsilon + R \epsilon dt_1 + R \epsilon^2 + R \epsilon^2 dt_1 + \cdots$$
$$= R + R dt_1 + R dt_1 \wedge dt_2 + R dt_1 \wedge dt_2 \wedge dt_3 + \cdots .$$

Due to the Bäcklund transformation, $\mathcal{M}^C_{elas}$ is topologically given as a telescopic type space related to these genera. Hence we have

$$H^*(\mathrm{for}(\mathcal{M}^C_{elas}), R) = \Lambda_R[dt_1, \epsilon].$$
Final Remark
Open problems

1. These relations are closely related to \( \log \frac{Z(p) - Z(q)}{p - q} \), which is also related to replicable functions in Monster group by John McKay. Investigate this fact!!!!

2. Show the explicit expression of quantized elastica or quantized elastica of genus \( g > 1 \) in terms of computer graphic and so on.

3. Show the degenerate limit from the quantized elasticas of \( g \) to \( g - 1 \).
Final Remark

Open problems

1. A quantized elastica in \((p, q)\)-dimensional Minkowski space with \(\text{so}(p, q)\) and generalized MKdV equation.

2. Willmore surface (Polynakov extrinsic string) and MNV hierarchy (M 1999),

3. A geometrical object expressed by **generalized Weierstrass representation** of submanifold Dirac operator (M 2008, 2009),

4. **Diff/SDiff** for a manifold which B. Khesin (**Arnold-Khesin**) considers, or fluid dynamics.
open problems

Since we partially have the hyperelliptic solutions of loop solitons (M 2002), we will consider the moduli space.


Thank you!!